Prof. Dr. Peter Bastian, Michal Tóth Submission date: 15 Nov 2021

IWR, Universität Heidelberg

Exercise 1 *Updating dune-npde*

As we update the *dune-npde* module during the semester, you need to get the current state before starting to solve a new programming exercise:

- Navigate to your *dune-npde* directory in a terminal
- Execute the commands

```
git stash
git pull
git stash pop
```

These *git* commands temporarily move your local changes to the stash, download the updates and apply your changes to the new version again.

(0 Points)

Exercise 2 *Properties of energy functional in discrete spring system*

In the lecture the equation for the total energy stored in the system at state u was derived

$$J^{(n)}(u) = J^{(n)}_{\text{el}} + J^{(n)}_{\text{f}} = \sum_{i=0}^{n} \frac{\kappa_i}{2}(\|u_{i+1} - u_i\|) - l_i)^2 - \sum_{i=1}^{n} u_i \cdot f_i$$

where $J^{(n)} : U \to \mathbb{R}$ and

$$U = \underbrace{\mathbb{R}^3 \times \mathbb{R}^3 \times \cdots \times \mathbb{R}^3}_{n+1 \text{ times}}.$$

This corresponds to a discrete approximation of the elastic and potential energy (see lecture for details).

Assume there exists $\varepsilon \in (0, 1)$ such that

$$\frac{\varepsilon}{2} \sum_{i=0}^{n} \|u_{i+1} - u_i\| \geq \sum_{i=0}^{n} l_i.$$

Show that the functional $J^{(n)}(u)$ is bounded from below, i.e.

$$\exists C \in \mathbb{R} : J^{(n)}(u) \geq C, \quad \forall u \in U. \tag{1}$$

To proof that proceed as follows:

1. At first show that:

$$J^{(n)}_{\text{el}} \geq \alpha \left(\sum_{i=0}^{n} \|u_{i+1} - u_i\| \right)^2 + \beta \qquad (\alpha, \beta > 0). \tag{2}$$

2. Futhermore, prove that:

$$\|u\| \leq \sqrt{n+2}\left(\sum_{i=0}^{n}\|u_{i+1}-u_i\|+\|u_0\|\right). \tag{3}$$

3. Finally, use both results to show that

$$J_{\text{el}}^{(n)} \geq \frac{\alpha^*}{2}\|u\|^2+\beta^*, \qquad (\alpha^*>0, \beta^* \in \mathbb{R})$$

The statement (1) is a combination of (2), (3) together with inequality for $J_{\text{f}}^{(n)}(u)$.

Helpful inequality:

$$\left(\sum_{i=1}^{n} a_i\right)^2 \leq n\sum_{i=1}^{n} a_i^2$$

(10 Points)

Exercise 3 *Analytical Solution For Heat Transfer Equation*

Consider the one-dimensional heat transfer equation

$$\partial_t u - \partial_x^2 u = 0$$

in the space-time domain

$$D^+ = \left\{(x,t) \in \mathbb{R}^2 | 0 < x < 1,\ 0 < t < \infty\right\}.$$

1. Show that the initial value problem with initial value $u(x,t)\big|_{t=0} = f$, $f \in C^1([0,1])$ and boundary condition $u(0,t) = u(1,t) = 0$ is solved by

$$u(x,t) = \sum_{n=1}^{\infty} \tilde{f}_n e^{-n^2\pi^2 t}\sin(n\pi x),$$

   where $\tilde{f}_n$ denotes the n-th fourier coefficient of f. In order to do this you can use separation of variables $u(x,t) = v(x)\cdot w(t)$.

2. Show that

$$\Phi = \frac{1}{\sqrt{4t}}e^{-\frac{x^2}{4t}}$$

   is a solution of heat trasfer equation.

(5 Points)

Exercise 4 *Heat Transport*

Consider the one-dimensional heat transport equation

$$\partial_t u - \partial_x^2 u = 0$$

on the space-time domain

$$D^+ = \left\{(x,t) \in \mathbb{R}^2 \,|\, 0 < x < 1,\ 0 < t < \infty\right\}.$$

The program used to solve this exercise can be found in *dune-npde/uebungen/uebung03* of your *dune-npde* module. It already calculates and prints the fourier coefficients

$$a_n := 2\int_0^1 f(x)\sin(2\pi n x)\,dx \qquad b_n := 2\int_0^1 f(x)\cos(2\pi n x)\,dx \qquad (N \geq n \geq 0)$$

of the function

$$f(x,t) = \frac{1}{\sqrt{4t}} e^{-\frac{(x-0.5)^2}{4t}}$$

for time $t_0 = 0.001$. The function `uniformintegration()` is used to determine the necessary integrals. This is allegedly done with a given precision, that can be set with parameter `accuracy`.

1. Describe how the function `uniformintegration()` works. Specify the circumstances under which you can actually estimate the quadrature error with `accuracy`.
2. The quadrature order of local gauss quadrature can be set in the configuration file *uebung03.ini*. Examine the convergence of occuring integrals. Does the convergence behaviour correspond to your expectatitons?
3. Implement a functor realizing the function

$$g(x,t) = \frac{b_0}{2} + \sum_{n=1}^{N} e^{-n^2 4\pi^2 (t-t_0)} \left(a_n \sin(n2\pi x) + b_n \cos(n2\pi x)\right).$$

4. Implement a functor that calculates

$$e(t) = \int_0^1 (g(x,t) - f(x,t))^2 dx$$

   using `uniformintegration()`. How does $e(t)$ change from $t = 0.001$ to $t = 0.02$? Create vtk files to visualize f and g for time step distance $\Delta t = 0.001$ and explain your observations.

(8 Points)