Prof. Dr. Peter Bastian, Michal Tóth Submission date: 31 Jan 2022

IWR, Universität Heidelberg

Note: Do not forget to update your dune installation as described in exercise sheet 2.

Exercise 1 *Crouzeix-Raviart A-posteriori error estimation*

Let $\Omega \subset \mathbb{R}^2$ be a domain with Lipschitz boundary and $\{\mathcal{T}_h\}_h$ be a family of conform and shape regular simplex triangulations with maximum edge size h. Let $\mathcal{F}_h$ denote the set of all edges of the mesh $\mathcal{T}_h$. For given $\mathcal{T}_h$ we define following spaces

$$P^1_{c,0} = \{v_h \in C^0(\Omega), \quad v_h\big|_{\partial\Omega} = 0, \quad \forall t \in \mathcal{T}_h : v_h\big|_t \in \mathbb{P}^1\}$$

and

$$P^1_{pt,0} = \{v_h \in L^1(\Omega), \quad \forall t \in \mathcal{T}_h : v_h\big|_t \in \mathbb{P}^1, \quad \forall e \in \mathcal{F}_h : \int_e [\![v_h]\!]_e = 0\},$$

where $[\![f]\!]_e$ is the jump of function f over edge e (if $e \subset \partial\Omega$ we define $f\big|_e = 0$). Only space $P^1_{c,0}$ is a susbspace of $H^1_0(\Omega)$.

For space $V_h := P^1_{pt,0}$ the bilinear form $a_h : V_h \times V_h \to \mathbb{R}$ is defined as

$$a_h(u_h, v_h) = \sum_{t\in\mathcal{T}_h} \int_t \nabla u_h \cdot \nabla v_h \, dx$$

together with the energy norm

$$\|v_h\|_{V_h} := \sqrt{a_h(v_h, v_h)}.$$

The function $u \in H^1_0(\Omega)$ and $u_h \in V_h$ are solution of

$$\forall v \in H^1_0(\Omega) : \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx.$$

and

$$\forall v_h \in V_h : \sum_{t\in\mathcal{T}_h} \int_t \nabla u_h \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx.$$

Show the a-posteriori error estimation:

$$\|u - u_h\|_{V_h} \le c\Big(\sum_{t\in\mathcal{T}_h} e_t(u_h, f)^2\Big)^{\frac{1}{2}} + 2 \inf_{v_h \in P^1_{c,0}} \|u_h - v_h\|_{V_h},$$

where the constant c depends only on the shape regularity of the mesh (independent of h) and use

$$e_t(u_h, f) = h_t \|f + \Delta u_h\|_{0,t} + \frac{1}{2} \sum_{e\in\partial t} h_e^{\frac{1}{2}} \|[\![\partial_n u_h]\!]\|_{0,e},$$

with h_e and h_t denoting the length of e and the longest edge in $\mathcal{T}_h$ respectively. **(6 Points)**

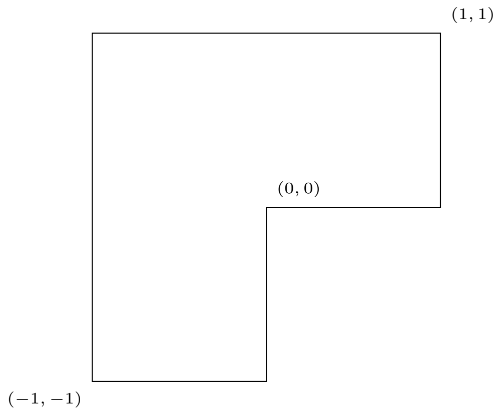
Exercise 2 *A-posteriori error estimation for Laplace equation*

In the lecture *a posteriori* error estimator for the second order elliptic boundary problem was derived. In this exercise we consider the Laplace equation with Dirichlet boundary conditions:

$$-\Delta u = 0, \quad x \in \Omega$$
$$u(x) = g(x), \quad x \in \partial\Omega$$

on a polygon domain $\Omega \subset \mathbb{R}^d$ (not necessary convex). We restrict our estimation to P^1 finite elements. Follow the lecture and derive the *a posteriori* error estimation for the error $e_h = u - u_h$. **(4 Points)**

Exercise 3 *Residual estimation*



In this exercise, we will solve reentrant corner problem using grid adaptation. We consider the Poisson problem in two space dimensions $\Delta u = 0$ on domain Ω (see picture) with the exact solution in polar coordinates

$$g(r, \varphi) = r^{\frac{2}{3}} \sin(\frac{2}{3}\varphi).$$

In *uebungen/uebung12* of your *dune-npde* module you can find a program that solves this problem for conform P_1 elements on the simplex mesh. Your task is to implement the *a-posteriori* error estimation for P_1 elements and use it to the local mesh adaptation.

1. The code is almost complete. You should only complete the implementation of the function `computeLocalError()` in the file *local_error.hh*, that should compute the residual error indicators η_t from the lecture.
2. The function `adaptGrid()` adapts the mesh dependent on values in vector `indicators`. You can decide between two strategies which cells to refine. Describe the difference between these two strategies.
3. Choose one of the strategies and compare the achieved accuracy (with respect to the number of degrees of freedom) to results using global mesh refinement.

(10 Points)