Multilevel Overlapping Domain Decomposition Methods with Spectral Coarse Spaces

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joint work with D. Kempf[1], L. Seelinger[2], R. Scheichl[1,2] and A. Strehlow[2]

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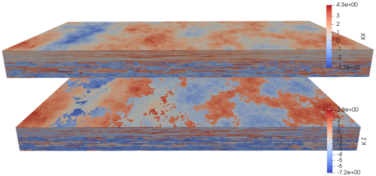
Outline

Section 1

Motivation

Groundwater Flow

$$
\begin{aligned}
-\nabla \cdot (K \nabla u) &= f && \text{in } \Omega,\\
u &= g && \text{on } \Gamma_D \subseteq \partial\Omega,\\
-(K\nabla u)\cdot \nu &= j && \text{on } \Gamma_N = \partial\Omega \setminus \Gamma_D.
\end{aligned}
$$



- Permeability tensor $K(x)$ **highly varying and/or anisotropic**
- Weighted symmetric interior penalty discontinuous Galerkin method[1]

$$
u_h \in V_h^{\mathrm{DG}} \quad : \quad a_h^{\mathrm{DG}}(u_h, v) = l_h^{\mathrm{DG}}(v) \qquad \forall v \in V_h^{\mathrm{DG}}
$$

$$
\begin{aligned}
a_h^{\mathbf{DG}}(u,v) = {} & \sum_{\tau \in \mathcal{T}_h} (K\nabla u, \nabla v)_{\mathbf{0},\tau} + \sum_{\gamma \in \mathcal{F}_h^I} \Big[\sigma_\gamma (\llbracket u \rrbracket, \llbracket v \rrbracket)_{\mathbf{0},\gamma} - (\{K\nabla u\}_\omega \cdot \nu_\gamma, \llbracket v \rrbracket)_{\mathbf{0},\gamma} - (\{K\nabla v\}_\omega \cdot \nu_\gamma, \llbracket u \rrbracket)_{\mathbf{0},\gamma} \Big] \\
& + \sum_{\gamma \in \mathcal{F}_h^D} \Big[\sigma_\gamma (u, v)_{\mathbf{0},\gamma} - ((K\nabla u)\cdot \nu_\gamma, v)_{\mathbf{0},\gamma} - ((K\nabla v)\cdot \nu_\gamma, u)_{\mathbf{0},\gamma} \Big]
\end{aligned}
$$

[1] *Ern, Stephansen, Zunino.* IMA Journal of Numerical Analysis, 29 (2008).

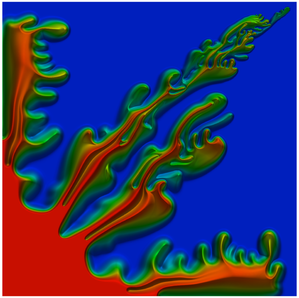
Unstable Flows in Porous Media

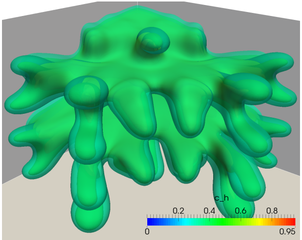
$$\nabla \cdot v = 0, \qquad v = -\frac{K}{\mu(c)}(\nabla p - \rho(c)g)$$

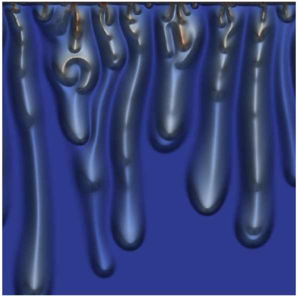
$$\partial_t(\Phi c) + \nabla \cdot (cv - D(v)\nabla c) = 0$$

Density driven flow, miscible discplacement

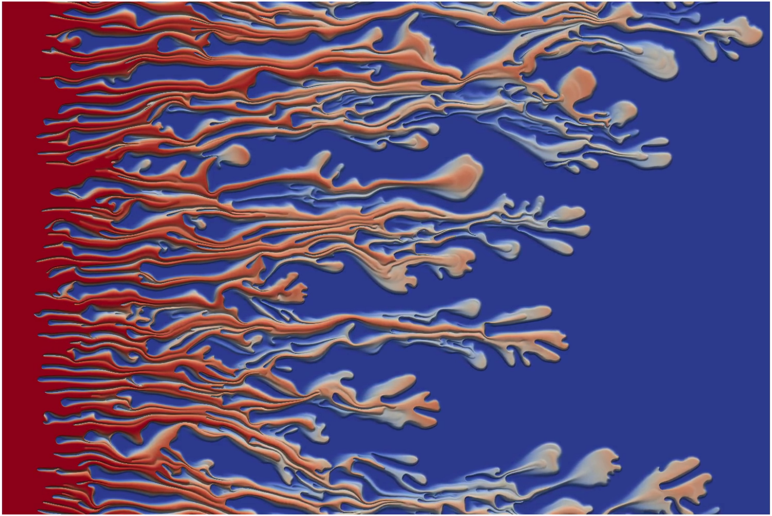
Can exploit high resolution schemes





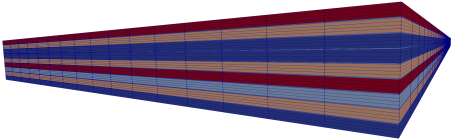


Viscous Fingering $Pe = 14400$, $M = 100$



Carbon Fibre Composites

$$
\begin{aligned}
-\nabla \cdot \sigma(u) &= f \\
\sigma(u) &= C : \epsilon(u) \\
\epsilon(u) &= \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)
\end{aligned}
$$

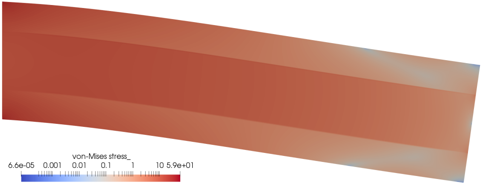


This example by L. Seelinger, A. Reinarz

- u displacement, σ Cauchy stress, ϵ strain, C stiffness
- Discretized with conforming $\mathbb{Q}_2$ serendipity elements
- Application to carbon fibre composites[2]:
  - anisotropic stiffness tensor C
  - discontinuous material properties
  - highly anisotropic meshes

[2] *Butler, Dodwell, Reinarz, Sandhu, Scheichl, Seelinger.* Computer Physics Communications, 249 (2020).

Reinforced Bulk Material



Example by
D. Kempf

- Two-dimensional bulk material reinforced by truss-beam elements[3]
- Discretized with cut-cell continuous Galerkin $\mathbb{P}_2$ / C^0 interior penalty discontinuous Galerkin method

	UMFPack	CG-SSOR		CG-GenEO		
#dof	T	#IT	T	#IT	T	S
3402	0.03	1229	0.8	30	0.44	3
13202	0.54	22691	64.3	35	1.92	14
822402	265.10			60	253.60	370

[3] *Hansbo, Larson, Larsson.* Lecture Notes in Computational Science and Engineering 121.

HPC FEM

- FEM for elliptic PDE leads to solving large sparse linear systems $Ax = b$
- Use iterative solvers $x^{k+1} = x^k + B(b - Ax^k)$ with preconditioner B
- **Matrix-based methods** assemble A in sparse matrix format
  - Memory bandwidth bound since $I \leq 1/4$ in $y = Ax$
  - **Robust preconditioners** B available, e.g. AMG, DD
- **Matrix-free methods** avoid assembling A to get compute-bound
  - Low-order FEM:
    - ★ Exploit structure: regular grids, constant coefficients
    - ★ Vectorization over several elements
  - High-order FE schemes: naive complexity $O(p^{2d}h^{-d})$ for $y = Ax$
    - ★ Complexity reduction by exploiting **tensor-product structure**: $O(p^{d+1}h^{-d})$
    - ★ Algorithm employs matrix-matrix products
    - ★ **Less flops are executed at a higher rate**
- **Our focus here:**
  - Coefficient and h-robust, matrix-based preconditioner B

Matrix-free vs. Matrix-based Operator Application

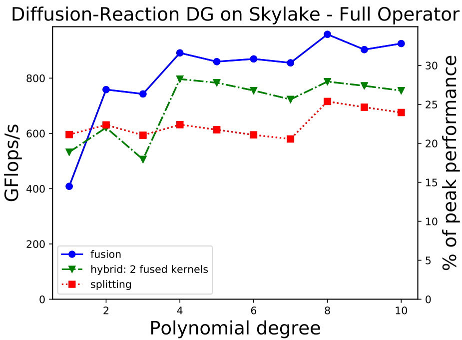
- Convection-Diffusion-Reaction Problem, var. permeability, axis-parallel grid
- PETSc matrix (25 GFLOP/s $\equiv$ at least 100 GByte/s)
- Node Performance on $2 \times$ Xeon E5-2698v3

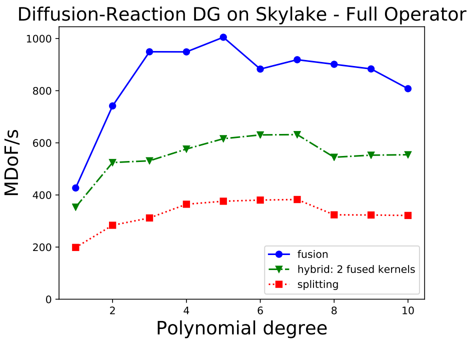
	Matrix-free		Matrix-based		Matrix Assembly	
p	$\frac{\text{DOF}}{\text{s}}$	$\frac{\text{GFLOP}}{\text{s}}$	$\frac{\text{DOF}}{\text{s}}$	$\frac{\text{GFLOP}}{\text{s}}$	$\frac{\text{DOF}}{\text{s}}$	$\frac{\text{GFLOP}}{\text{s}}$
1	1.80×10^8	126	2.06×10^8	24.3	1.60×10^7	345
2	4.17×10^8	282	6.42×10^7	27.3	8.71×10^6	371
3	6.00×10^8	376	2.69×10^7	29.8	4.66×10^6	368
4	6.76×10^8	436	9.54×10^6	23.5	2.57×10^6	301
5	7.20×10^8	467	4.58×10^6	23.3	1.93×10^6	307
6	7.22×10^8	481	2.31×10^6	23.5	1.21×10^6	231
7	7.07×10^8	481	1.46×10^6	30.3	6.65×10^5	143
8	6.80×10^8	476	6.14×10^5	24.1	8.74×10^5	198
9	6.59×10^8	470	2.98×10^5	26.2	7.01×10^5	133

Results by Steffen Müthing

Reaction Diffusion DG - Skylake

- Full Diffusion Tensor, SIPG DG discretization
- Measure matrix-free Jacobian application





- $(u, \nabla u) \times$ 2 qp, $(u, \partial_1 u) \times$ 4 qp + $(\partial_2 u, \partial_3 u) \times$ 4 qp, $u \times$ 8 qp, $\partial_i u \times$ 8 qp
- Results by Dominic Kempf

Let's focus

on efficient parallel iterative methods for

solving linear systems $\quad Ax = b$

where

- $A \in \mathbb{R}^n$ is **symmetric and positive definite**
- A arises from the **discretization of** (systems of) **PDEs**
- with **highly varying and/or anisotropic coefficients/meshes**
- n **is very large**, e.g. $n \approx 10^8 \ldots 10^{12}$
- the number of available computers P **is very large**, e.g. $P \approx 10^5 \ldots 10^6$
- S.p.d. case allows rigorous theory, concept is more general

Robust Parallel Iterative Solvers

- Algebraic multilevel methods (as early as 1982)
  - Ruge-Stüben, Agglomeration, AMGe, AMLI
  - Spectral AMGe, *Chartier, et al.*, SIAM J. Sci. Comput., 25(1), 2006
- Two-level overlapping and nonoverlapping domain decomposition (DD)
  - Not coefficient-robust in their standard form
- Two-level spectral DD methods
  - *Galvis, Efendiev*, Multiscale Model. Simul. 8 (2010) 1461–1483
  - *Efendiev et al.* M2AN 46 (2012) 1175-1199
  - Multilevel variant: *Willems*, SIAM Journal on Numerical Analysis, 52 (2014)
  - **GenEO: *Spillane et al.*, Numerische Mathematik, 126, 2014**
  - SORAS-GenEO-2: *Haferssas, Jolivet, Nataf*, SIAM SISC 39(4) 2017
  - Algebraic spsd splitting: *Daas, Grigori*, SIAM Mat. Ana. & Appl. 40(1) 2019
  - Multilevel spaces: *Daas et al.*, hal-02151184 2020
- Spectral coarse spaces . . .
  - . . . used for preconditioning
  - . . . used as multiscale method

Why two levels are not sufficient . . .

- Supercomputers have many cores. HAWK@HLRS: 720896 / 2GB
- Dimension of coarse problem is $\sim P$
- More subdomains than cores (oversubscription) is attractive
- Limits P to around 10^4 or even less
- Three or four levels are often sufficient
  - (A/G)MG: #subdomains $\sim n$, coarsening factor 2^d
  - MLDD: #subdomains $\sim P$, agressive coarsening factor ≈ 100

New aspects in this work:

- Multilevel spectral DD as subspace correction method
- Theory for additive V-cycle (solve once in each subdomain)
- Theory for discontinuous Galerkin discretization
- Several options for the rhs of the eigenproblems

Section 2

Multilevel Spectral DD as Subspace Correction Method

Abstract Subspace Correction Methods

Consider the **discrete variational problem**

$$u_h \in V_h : \qquad a(u_h, v) = l(v) \qquad \forall v \in V_h,$$

V_h: finite element space, a: symmetric, coercive bilinear form, l linear form

Split V_h into P (possibly overlapping) subspaces

$$\boxed{V_h = V_{h,1} + \ldots + V_{h,P}.}$$

Parallel (additive) subspace iteration

$$u_h^{k+1} = u_h^k + \omega \sum_{i=1}^{P} w_i^k, \qquad 0 < \omega \in \mathbb{R}$$

$$w_i^k \in V_{h,i} : \qquad a(w_i^k, v) = l(v) - a(u_h^k, v) \qquad \forall v \in V_{h,i}.$$

Sequential subspace correction and hybrid variants are possible

Multilevel Domain Decomposition Splitting

I: Multilevel hierarchy of finite element spaces (vertical)

$$V_{h,0} \subset V_{h,1} \subset \ldots \subset V_{h,l} \subset \ldots V_{h,L} = V_h$$

II: Split each level into P_l subspaces related to subdomains (horizontal)

$$V_{h,l} = \sum_{i=1}^{P_l} V_{h,l,i}$$

- With $P_0 = 1$ this yields a multilevel DD splitting:

$$\boxed{V_h = V_{h,0} + \sum_{l=1}^{L} \sum_{i=1}^{P_l} V_{h,l,i}}$$

Q: How to define $V_{h,l}$ and $V_{h,l,i}$?

Algebraic Formulation

Implementation: introduce **basis representations** of the subspaces

$$V_h = \text{span}\{\phi_1, .., \phi_n\}, \quad V_{h,l,i} = \text{span}\{\phi_{l,i,1}, .., \phi_{l,i,n_{l,i}}\}, \quad \phi_{l,i,j} = \sum_{k=1}^{n} (R_{l,i})_{j,k}\, \phi_k.$$

Parallel subspace correction in algebraic form

Expanding $u_h^k = \sum_{j=1}^n \left(x^k\right)_j \phi_j$ gives

$$x^{k+1} = x^k + \omega \left(\underbrace{R_0^T A_0^{-1} R_0 + \sum_{l=1}^{L} \sum_{i=1}^{P_l} R_{l,i}^T A_{l,i}^{-1} R_{l,i}}_{\text{preconditioner } B} \right) (b - Ax^k)$$

with $(A)_{r,s} = a(\phi_s, \phi_r)$, $A_{l,i} = R_{l,i} A R_{l,i}^T$, $R_0 = R_{0,1}$, $A_0 = R_0 A R_0^T$.

Convergence

- Number of iterations needed is proportional to (square root of) the spectral condition number

$$\kappa(BA) = \frac{\lambda_{\mathrm{max}}(BA)}{\lambda_{\mathrm{min}}(BA)}$$

- Below we are interested in upper and lower bounds

$$\gamma \leq \lambda_{\mathrm{min}}(BA) \leq \kappa(BA) \leq \lambda_{\mathrm{max}}(BA) \leq \Gamma$$

which implies $\kappa(BA) \leq \Gamma/\gamma$

- These bounds are proven on the function level using properties of bilinear forms and finite element spaces

Multilevel Domain Decomposition

- $\mathcal{T}_h$ is a finite element mesh for the domain Ω
- On each level l decompose $\mathcal{T}_h$ into P_l **overlapping** subsets:

$$\mathcal{T}_{h,l,i} \subset \mathcal{T}_h, \quad 1 \leq i \leq P_l, \qquad\qquad \bigcup_{i=1}^{P_l} \mathcal{T}_{h,l,i} = \mathcal{T}_h,$$

- defining the subdomains

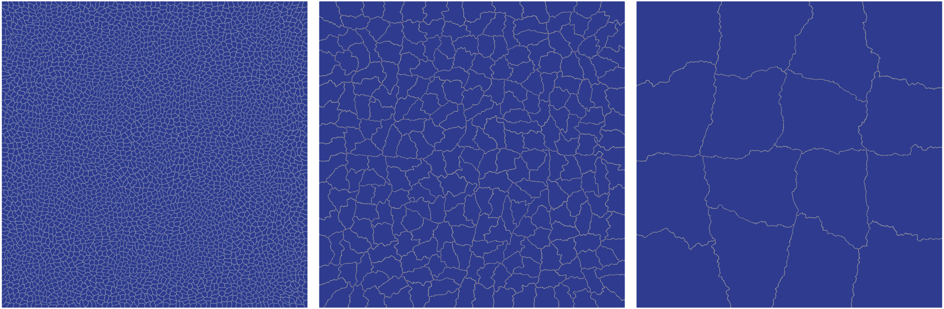
$$\Omega_{l,i} = \text{Interior}\left(\bigcup_{\tau \in \mathcal{T}_{h,l,i}} \bar{\tau} \right)$$

- Decomposition is **hierarchic**:

$$\mathcal{T}_{h,l,i} = \bigcup_{k \in J_{l,i}} \mathcal{T}_{h,l+1,k}, \qquad 0 \leq l < L, \qquad P_0 = 1, P_l < P_{l+1}$$

$$\bigcup_{i=1}^{P_l} J_{l,i} = \{1, \ldots, P_{l+1}\} \text{ describes union of subdomains}$$

Multilevel Domain Decomposition: 2D Example



$P_3 = 4096$ $P_2 = 256$ $P_1 = 16$

Domain decomposition hierarchy generated with ParMetis

Coarsest level not shown

Multilevel Spectral Finite Element Spaces

- $r_{l,i} v = v|_{\Omega_{l,i}}$ restriction operators, $e_{l,i}$ zero extension operators
- $\chi_{l,i}$ partition of unity operators: $\sum_{i=1}^{P_l} e_{l,i}\chi_{l,i} r_{l,i} v = v$, $\chi_{l,i} v_{l,i}|_{\partial\Omega_{l,i}\cap\Omega} = 0$
- On level L set $V_{h,L} = V_h$ defined on mesh $\mathcal{T}_h$
- For $l = L, \ldots, 1$ construct $V_{h,l-1}$ from $V_{h,l}$ as follows
  - For each subdomain $i = 1, \ldots, P_l$ define the spaces

$$\begin{aligned} V_{h,l,i} &= \left\{ v \in V_{h,l} : \operatorname{supp} v \subseteq \overline{\Omega}_{l,i} \right\} \subset V_{h,l} && \text{used in subspace correction} \\ \bar{V}_{h,l,i} &= \{ r_{l,i} v : v \in V_{h,l} \} && \text{auxiliary space} \end{aligned}$$

  - In each subdomain $i = 1, \ldots, P_l$, solve a generalized eigenproblem

$$w_{l,i,k} \in \bar{V}_{h,l,i} : \qquad \bar{a}_{l,i}(w_{l,i,k}, v) = \lambda_{l,i,k} \bar{b}_{l,i}(w_{l,i,k}, v) \qquad \forall v \in \bar{V}_{h,l,i}$$

    positive semi-definite BLFs $\bar{a}_{l,i}$ and $\bar{b}_{l,i}$ to be detailled below
  - Then for a given threshold $0 < \eta \in \mathbb{R}$ set

$$V_{h,l-1} = \bigoplus_{i=1}^{P_l} \operatorname{span} \{ \phi_{l,i,k} : \phi_{l,i,k} = e_{l,i}\chi_{l,i} w_{l,i,k} \wedge \lambda_{l,i,k} < \eta \}$$

Section 3

Convergence Theory

Abstract Subspace Correction Theory

Stable splitting

The multilevel splitting is called C_0-stable if there exists $C_0 > 0$ and for each $v \in V_h$ a decomposition $v = v_0 + \sum_{l=1}^{L} \sum_{i=1}^{P_l} v_{l,i}$, such that

$$a(v_0, v_0) + \sum_{l=1}^{L} \sum_{i=1}^{p} a(v_{l,i}, v_{l,i}) \leq C_0\, a(v, v).$$

Levelwise Coloring

$\exists k_0 \in \mathbb{N}$ and maps $c_l : \{1, \ldots, P_l\} \to \{1, \ldots, k_0\}$, such that for each $l > 0$

$$i \neq j \wedge c_l(i) = c_l(j) \qquad \Rightarrow \qquad a(v_{l,i}, v_{l,j}) = 0 \quad \forall v_{l,i} \in V_{h,l,i}, \forall v_{l,j} \in V_{h,l,j}.$$

Theorem 1

Under these assumptions

$$\kappa(BA) \leq (1 + k_0 L) C_0.$$

Abstract Spectral Subspace Correction Theory

A1 (Triangle inequality under the square). $\exists a_0 > 0$ indep. of $l > 0, P_l$ s. t.

$$\left\| \sum_{i=1}^{P_l} v_{l,i} \right\|_a^2 \leq a_0 \sum_{i=1}^{P_l} \|v_{l,i}\|_a^2 , \qquad \forall v_{l,i} \in V_{h,l,i}.$$

A2 (Symmetric positive semidefinite splitting). $\exists b_0 > 0$ such that for $l > 0$ the bilinear forms $\bar{a}_{l,i}$ satisfy

$$\sum_{i=1}^{P_l} |r_{l,i} v_l|_{\bar{a}_{l,i}}^2 \leq b_0 \; \|v_l\|_a^2 , \qquad \forall v_l \in V_{h,l}.$$

A3 (Local stability). $\exists C_1 > 0$ and for each $v_l \in V_{h,l}$, $l > 0$, a decomposition $v_l = v_{l-1} + \sum_{i=1}^{P_l} v_{l,i}$ such that

$$\|v_{l,i}\|_a^2 \leq C_1 \, |r_{l,i} v_l|_{\bar{a}_{l,i}}^2 , \qquad 1 \leq i \leq P_l.$$

This two-level splitting can be extended to a multilevel splitting.

Stable Splittings

Lemma 2 (Two-level Stable Splitting)

Let A1-A3 be satisfied for all levels. Then, for $l > 0$ exists a decomposition $v_l = v_{l-1} + \sum_{i=1}^{P_l} v_{l,i}$ such that

$$\|v_{l-1}\|_a^2 + \sum_{i=1}^{P_l} \|v_{l,i}\|_a^2 \leq (2 + b_0 C_1(1 + 2a_0)) \|v_l\|_a^2 .$$

Lemma 3 (Multilevel Stable Splitting)

Let A1-A3 be satisfied for all levels. Then $v = v_0 + \sum_{l=1}^{L} \sum_{i=1}^{P_l} v_{l,i}$ satisfies

$$\|v_0\|_a^2 + \sum_{l=1}^{L} \sum_{i=1}^{P_l} \|v_{l,i}\|_a^2 \leq C^L \left(1 + \frac{b_0 C_1}{C - 1}\right) \|v\|_a^2$$

with $C = 2(1 + a_0 b_0 C_1)$. (This is due to $\|v_{l-1}\|_a \leq C\|v_l\|_a$).

Generalized Eigenvalue Problem I

- **Local stability A3** is ensured via the GEVP:

$$w_{l,i,k} \in \bar{V}_{h,l,i}: \qquad \bar{a}_{l,i}(w_{l,i,k}, v) = \lambda_{l,i,k}\bar{b}_{l,i}(w_{l,i,k}, v) \qquad \forall v \in \bar{V}_{h,l,i}$$

  $\bar{a}_{l,i}, \bar{b}_{l,i}$ symmetric positive semi-definite with $\ker \bar{a}_{l,i} \cap \ker \bar{b}_{l,i} = \{0\}$.

- Spectrum in subdomain i on level l has the form

$$0 = \underbrace{\lambda_1 = ... = \lambda_r}_{r=\dim \ker \bar{a}_{l,i}} < \lambda_{r+1} \leq ... \leq \lambda_{n-s} < \underbrace{\lambda_{n-s+1} = ... = \lambda_n}_{s=\dim \ker \bar{b}_{l,i}} = +\infty,$$

- Eigenfunctions $w_{l,i,k}$ can be chosen $\bar{b}_{l,i}$-orthonormal and $\bar{a}_{l,i}$-orthogonal

Generalized Eigenvalue Problem II

- Define projection $\Pi_{l,i,m} : \bar{V}_{h,l,i} \to \bar{V}_{h,l,i,m} = \text{span}\{w_{l,i,k} : 1 \le k \le m\}$

$$\Pi_{l,i,m} v = \sum_{k=1}^{m} \bar{b}_{l,i}(v, w_{l,i,k}) w_{l,i,k}.$$

- $\Pi_{l,i,m}$ is $\bar{a}_{l,i}$-stable:

$$|\Pi_{l,i,m} v|^2_{\bar{a}_{l,i}} \le |v|^2_{\bar{a}_{l,i}}, \qquad |(\mathrm{I} - \Pi_{l,i,m})v|^2_{\bar{a}_{l,i}} \le |v|^2_{\bar{a}_{l,i}}, \qquad v \in \bar{V}_{h,l,i}.$$

- If $m \ge \dim(\ker \bar{a}_{l,i})$ one can show

$$|(\mathrm{I} - \Pi_{l,i,m})v|^2_{\bar{b}_{l,i}} \le \lambda^{-1}_{m+1} |(\mathrm{I} - \Pi_{l,i,m})v|^2_{\bar{a}_{l,i}}, \qquad v \in \bar{V}_{h,l,i}.$$

- **The two-level splitting in A3 is then defined as**

$$v_l = \underbrace{\sum_{i=1}^{P_l} e_{l,i}\chi_{l,i}\Pi_{l,i,m(\eta)} r_{l,i} v_l}_{v_{l-1}} + \sum_{i=1}^{P_l} \underbrace{e_{l,i}\chi_{l,i}(\mathrm{I} - \Pi_{l,i,m(\eta)}) r_{l,i} v_l}_{v_{l,i}}$$

Application to Discontinuous Galerkin

Use the weighted SIPG method by *Di Pietro, Ern, Guermond, (2008).*
The BLF has volume and face contributions:

$$a_h^{\mathrm{DG}}(u,v) = \sum_{\tau\in\mathcal{T}_h} a_\tau(u,v) + \sum_{\gamma\in\mathcal{F}_h^I} a_\gamma^I(u,v) + \sum_{\gamma\in\mathcal{F}_h^D} a_\gamma^D(u,v).$$

For the subdomains set

$$\bar{a}_{l,i}(u,v) = \sum_{\tau\in\mathcal{T}_{h,l,i}} a_\tau(u,v) + \sum_{\gamma\in\mathcal{F}_{h,l,i}^I} a_\gamma^I(u,v) + \sum_{\gamma\in\mathcal{F}_{h,l,i}^D} a_\gamma^D(u,v),$$

where

$$\mathcal{F}_{h,l,i}^I = \{\gamma\in\mathcal{F}_h^I : \gamma\subset\Omega_{l,i}\}, \qquad \mathcal{F}_{h,l,i}^D = \{\gamma\in\mathcal{F}_h^D : \gamma\subset\partial\Omega_{l,i}\}.$$

Then A1, A2 are satisfied with constants independent of P and K

A3 for Discontinuous Galerkin

For the DG method consider three different choices for $\bar{b}_{l,i}$:

$$\bar{b}^1_{l,i}(u,v) = \bar{a}_{l,i}\left(\chi_{l,i}u, \chi_{l,i}v\right) = a^{\mathrm{DG}}_h(\chi_{l,i}u, \chi_{l,i}v),$$
$$\bar{b}^2_{l,i}(u,v) = \mathring{a}_{l,i}\left(\chi_{l,i}u, \chi_{l,i}v\right), \quad \text{original GenEO}$$
$$\bar{b}^3_{l,i}(u,v) = \bar{a}_{l,i}\left((\mathrm{I}-\chi_{l,i})u, (\mathrm{I}-\chi_{l,i})v\right),$$

Lemma 4

If eigenvalues are picked according to a threshold $\eta > 0$, A3 is satisfied with

$$C^1_1 = \eta^{-1}, \qquad C^2_1 = 1 + \eta^{-1}, \qquad C^3_1 = 2\left(1+\eta^{-1}\right).$$

$$\mathring{a}_{l,i}(u,v) = \sum_{\tau \in \mathring{\mathcal{T}}_{h,l,i}} a_\tau(u,v) + \sum_{\gamma \in \mathring{\mathcal{F}}^I_{h,l,i}} a^I_\gamma(u,v) + \sum_{\gamma \in \mathring{\mathcal{F}}^D_{h,l,i}} a^D_\gamma(u,v)$$

corresponding to $\mathring{\Omega}_{l,i} = \{x \in \Omega_{l,i} : x \text{ is shared by other subdomains}\} \subset \Omega_{l,i}$.

Section 4

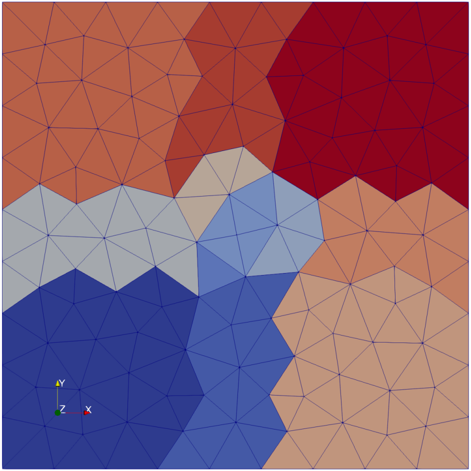
Numerical Results

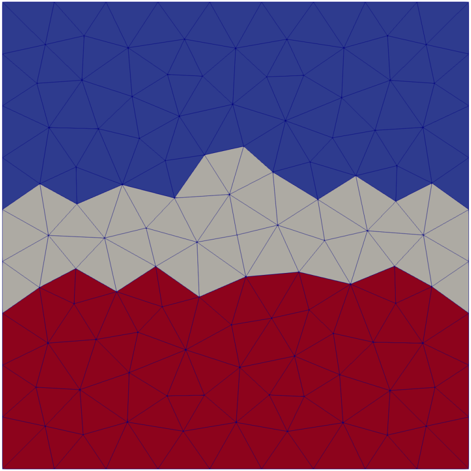
Implementation Details

- Code is **sequential**
- Parallel execution time and speedup are **estimated**
- **Fully algebraic implementation**
  - Patch stiffness matrices required on level L
  - Subdomain Neumann matrices would also suffice
  - Algebraic SPSD splittings have been devised
- ParMetis for mesh partitioning
- Arpack in symmetric shift-invert mode used for eigenproblems on all levels
- UMFPack and Cholmod used for subdomain problems
- All integrated in Dune[4] software framework
- 10^{-8} reduction of initial residual norm in all examples

[4] `www.dune-project.org`

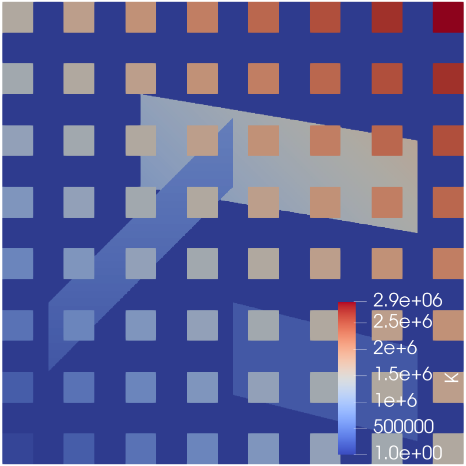
Patch-based Assembly

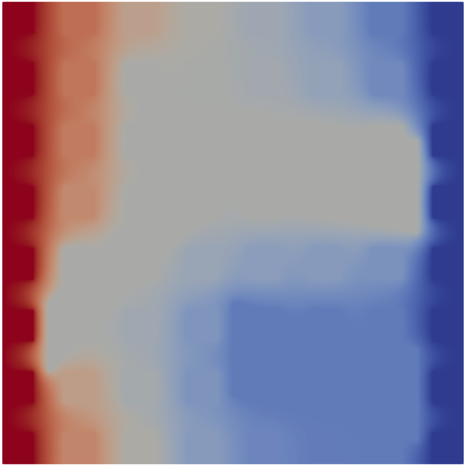




Islands Problem

Scalar diffusion problem with isotropic permeability field





h Scaling in Two-level Method

$\mathbb{Q}_1$ CGFE, 16 subdomains, $\eta = 0.15$, varying overlap δ

Problem	homogeneous				Islands			
	$\delta \sim h$		$\delta \sim H$		$\delta \sim h$		$\delta \sim H$	
h^{-1}	# IT	n_0	# IT	n_0	# IT	n_0	# IT	n_0
320	30	56	30	56	31	65	31	65
640	29	114	32	50	27	103	30	59
1280	27	236	25	54	27	240	25	78
2560	26	488	26	54	25	481	24	55

- Size of coarse space n_0 depends on overlap δ
- Convergence independent of coefficient K

p Scaling in Two-level Method

DG FE in 2d, 384^2 elements, 256 subdomains, varying polynomial degree p

		$\eta = 0.15$					$n_{ev} = 20$				
		$\delta = 2h$		δ var			$\delta = 2h$		δ var		
p	n_1	#IT	n_0	δ	#IT	n_0	#IT	n_0	δ	#IT	n_0
1	589824	28	1457	2	28	1457	18	5120	2	18	5120
2	1327104	21	3171	3	22	1901	19	5120	3	18	5120
3	2359296	20	5026	3	21	2991	20	5120	3	19	5120
4	3686400	18	8217	4	21	3322	21	5120	4	20	5120
5	5308416	17	13596	4	21	5078	23	5120	4	21	5120
6	7225344	17	17029	5	22	5234	24	5120	5	22	5120

- Even with fixed number of eigenvalues per subdomain and fixed overlap there is only mild increase in iteration numbers with polynomial degree

Islands 2d Weak Scaling

Fixed #dof/subdomain, $\mathbb{Q}_1$ conforming finite elements, $\delta = 3h$, $\eta = 0.3$

subdomains	64	256	1024	4096	16384
levels			degrees of freedom		
finest total	410881	1640961	6558721	26224641	104878081
n_0 2 levels	306	1348	5523	22673	91055
n_0 3 levels		130	431	1319	3890
n_0 4 levels			207	436	891
levels			#IT		
2	25	26	27	26	26
3		32	31	31	33
4			40	38	38

Conjecture: $\kappa(BA) = O(L^2)$

Islands 3d Strong Scaling

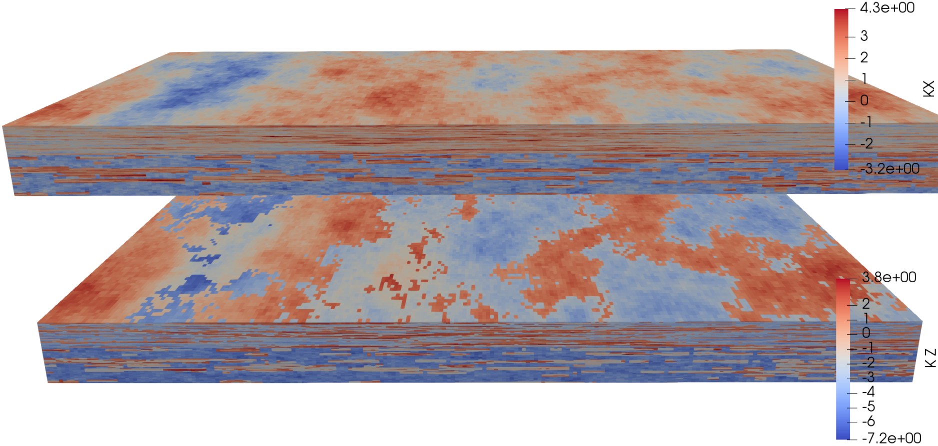
FV discretization, fixed problem size 320^3 mesh, 32768000 degrees of freedom

P_L	P_{L-1}	#IT	n_0	T_{seq}	T_{par}	$T_{i,min}$	$T_{i,max}$	T_{coarse}
two level method, varying overlap δ								
512	1	12	7680	63613	191.3	70.3	176.2	0.47
1024	1	12	15360	35817	58.4	18.2	49.8	1.3
2048	1	14	30720	18781	25.2	4.9	13.2	5.1
4096	1	13	61441	19982	33.5	2.2	7.0	20.1
three level method								
4096	32	15	1387	21168	55.9	9.8	42.3	0.27
4096	64	15	1817	20725	27.7	2.5	15.1	0.18
4096	128	16	2569	20549	18.4	0.59	6.2	0.15

- More subdomains are faster sequentially
- Three level method is faster than two-level method in parallel

SPE10 Problem

Scalar diffusion problem with strong heterogeneity and anisotropy



SPE10 Results

Overlap $\delta = 2h$, GMRES+MRAS cycle

P_L, P_{L-1}	CG, $n_L = 9124731$ #IT	n_0	T_{par}	CCFV, $n_L = 8976000$ #IT	n_0	T_{par}	DG, $n_L = 8976000$ #IT	n_0	T_{par}
two levels, $\eta = 0.3$									
256	24	7237	133.8	25	7502	49.2	22	8366	233.5
512	24	9830	60.4	23	10600	21.6	21	13690	124.9
1024	28	21881	22.3	25	25753	12.3	24	31637	42.4
2048	25	29023	15.7	25	35411	11.2	25	46844	36.7
three levels, no coarse overlap, $\eta = 0.3$									
256, 16	29	1222	151.4	29	1364	70.3	31	1683	273.2
512, 16	27	1228	77.8	28	1446	47.4	28	1762	186.8
1024, 32	36	3145	46.3	34	3487	47.3	33	5476	231.4
2048, 32	31	3120	40.1	35	3421	49.9	36	5359	204.8

- Method works equally well for different discretization schemes

3d Carbon Fibre Composites

- Linear elasticity, $\mathbb{Q}_2$ serendipity elements
- 9 ply layers and 8 interface layers
- $256 \times 64 \times 52$ mesh, 10523067 degrees of freedom
- 1024 subdomains
- GMRES+MRAS cycle

	subdomains				max dofs/subdomain				
levels	3	2	1	0	3	2	1	0	#IT
2			1024	1			28791	21565	13
3		1024	32	1		28791	1260	914	31
4	1024	128	16	1	28791	546	515	273	35

Conclusions

- Extended spectral DD methods to more than two levels
- Generalized GenEO theory to DG discretization schemes
- Fully additive, robust and scalable preconditioner for SPD problems
- Improvements over 2 level DD in $d = 3$, $P \geq 4096$, $n > 30 \cdot 10^6$

Many research opportunities remain:

- Improve the C^L factor in the estimate
- Better understand eigenvalue distribution
- High-performance, scalable, parallel implementation
- Non-symmetric problems

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Thank You for your attention!