Finite Elements (WS 2017) **Exercise 3**

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Exercise 1 *Tricomi Equation*

Consider the Tricomi equation

$$\partial_y^2 u + y\partial_x^2 u = 0$$

for a scalar-valued function u on the domain

$$\Omega = [-1,1] \times [-1,1].$$

Determine which class of equation it is.

This nonlinear equation can be used to model an object travelling at supersonic speed.

(2 Points)

Exercise 2 *Analytical Solution For Heat Transfer Equation*

Consider the one-dimensional heat transfer equation

$$\partial_t u - \partial_x^2 u = 0$$

in the space-time domain

$$D^+ = \left\{(x,t) \in \mathbb{R}^2 | 0 < x < 1,\ 0 < t < \infty\right\}.$$

1. Show that the initial value problem with initial value $u(x,t)\big|_{t=0} = f$, $f \in C^1([0,1])$ and boundary condition $u(0,t) = u(1,t) = 0$ is solved by

$$u(x,t) = \sum_{n=1}^{\infty} \tilde{f}_n e^{-n^2\pi^2 t} \sin(n\pi x),$$

   where $\tilde{f}_n$ denotes the n-th fourier coefficient of f. In order to do this you can use separation of variables $u(x,t) = v(x) \cdot w(t)$.

2. Show that

$$\Phi = \frac{1}{\sqrt{4t}} e^{-\frac{x^2}{4t}}$$

   is a solution of heat trasfer equation.

(5 Points)

Exercise 3 *Riesz Theorem (constructive proof)*

1. Find a constructive proof of Riesz Theorem:

   Let $(V, (.,.)_V)$ be a real Hilbert space and $v' \in V'$ an arbitrary linear form on V. Then there exists a unique $u \in V$ such that

   $$\langle v', w \rangle_{V',V} = (u, w)_V \qquad \forall w \in V.$$

   Moreover, $\|v'\|_{V'} = \|u\|_V$.

   Hints:

   (a) First prove the uniqueness (under the assumption of an existence) of u.

   (b) Let $M = \{w \in V | \langle v', w \rangle_{V',V} = 0\}$. Show that $M^\perp$ is a one-dimensional subspace of V (or $v' = 0$ holds) and that $V = M \oplus M^\perp$ holds.

   (c) Show that for $z \in M^\perp$ the vector u is given by

   $$u = \frac{\langle v', z \rangle_{V',V}}{\|z\|_V^2} z.$$

2. After prooving Riesz Theorem, show the second part:
   The map $\tau : V' \to V$ mapping $v' \in V'$ to the corresponding $u \in V$ is linear and an isometry, i.e. $\|\tau v'\|_V = \|v'\|_{V'}$.

(6 Points)