Finite Elements (WS 2017) **Exercise 4**

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Exercise 1 *Weak differentiability*

Continuous, piecewise-smooth functions in 1D are weakly differentiable (see example 5.31 in the lecture notes). The continuity of the function is crucial.

Consider the function

$$f(x) = \begin{cases} -1 & x \in (-1, 0] \\ 1 & x \in (0, 1) \end{cases}$$

and show that the weak derivative of f does not exist. **(2 Points)**

Exercise 2 *Projections*

Let Y be a subspace of a normed vector space X. An operator $P : X \to X$ is said to be a projection on Y if

$$P^2 = P \quad \text{and} \quad \mathrm{Range}(P) = Y.$$

Show the following:

1. P is a projection if and only if $P : X \to Y$ and $P = I$ on Y.
2. If P is a projection, then $X = \mathrm{Ker}(P) \oplus \mathrm{Range}(P)$, where $\oplus$ denotes a direct sum.

(4 Points)

Exercise 3 *Operators on Hilbert space*

Let H be a Hilbert space and Y a closed subspace of H. Define the map $P : H \to Y$ for each $v \in H$ as

$$\forall y \in Y : (P(v), y) = (v, y).$$

Let us prove that:

1. Operator P is linear and continuous.
2. For $v \in H$ it holds
$$\|P(v) - v\| = \min_{y \in Y} \|y - v\|.$$

Hint: Apply *Lax-Milgram Theorem* and *Characterization Theorem*.

(5 Points)

Exercise 4 *Heat Transport*

Consider the one-dimensional heat transport equation

$$\partial_t u - \partial_x^2 u = 0$$

on the space-time domain

$$D^+ = \left\{(x,t) \in \mathbb{R}^2 | 0 < x < 1,\ 0 < t < \infty\right\}.$$

The program used to solve this exercise can be found in *dune-npde/uebungen/uebung03* of your *dune-npde* module. It already calculates and prints the fourier coefficients

$$a_n := 2\int_0^1 f(x)\sin 2\pi nx\,dx \qquad b_n := 2\int_0^1 f(x)\cos 2\pi nx\,dx \qquad (N \geq n \geq 0)$$

of the function

$$f(x,t) = \frac{1}{\sqrt{4t}} e^{-\frac{(x-0.5)^2}{4t}}$$

for time $t_0 = 0.001$. The function `uniformintegration()` is used to determine the necessary integrals. This is allegedly done with a given precision, that can be set with parameter `accuracy`.

1. Describe how the function `uniformintegration()` works. Specify the circumstances under which you can actually estimate the quadrature error with `accuracy`.
2. The quadrature order of local gauss quadrature can be set in the configuration file *uebung03.ini*. Examine the convergence of occuring integrals. Does the convergence behaviour correspond to your expectatitons?
3. Implement a functor realizing the function

$$g(x,t) = \frac{b_0}{2} + \sum_{n=1}^{N} e^{-n^2 4\pi^2 (t-t_0)} \left(a_n \sin n2\pi x + b_n \cos n2\pi x\right).$$

4. Implement a functor that calculates

$$e(t) = \int_0^1 (g(x,t) - f(x,t))^2 dx$$

   using `uniformintegration()`. How does $e(t)$ change from $t = 0.001$ to $t = 0.02$? Create vtk files to visualize f and g for time step distance $\Delta t = 0.001$ and explain your observations.

(8 Points)