

Exercise 1 H^1 functions

Let $\Omega \subset \mathbb{R}^2$ be the unit cube, $\Omega = [0, 1] \times [0, 1]$.

1. For which α is the function in polar coordinates

$$f(r, \varphi) = r^\alpha \sin(\alpha\varphi) \quad (1)$$

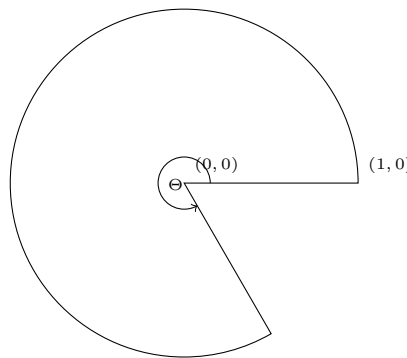
from space $H^1(\Omega)$?

2. The function (1) is a special form of the harmonic function

$$g(r, \varphi) = r^{\frac{\pi}{\Theta}} \sin\left(\frac{\pi}{\Theta}\varphi\right).$$

Show that g is harmonic, that means $\Delta g = 0$.

The Laplace-Problem $\Delta u = 0$ should be solved on a domain Ω (see figure). What Dirichlet conditions must be set such that g is a solution? Write them down explicitly.



(4 Points)

Exercise 2 Discontinuity of H^1 -functions in 2D and 3D

1. Consider the domain $\Omega = B(0, R) \subset \mathbb{R}^2$, where

$$B(0, R) = \{x \in \mathbb{R}^2 \mid \|x\| < R\}, \quad 0 < R < \frac{1}{e}.$$

Show in detail that the function

$$f(x) = \ln\left(\ln\left(\frac{1}{r(x)}\right)\right), \quad r(x) = \left(\sum_{i=1}^2 x_i^2\right)^{\frac{1}{2}}$$

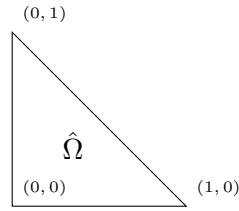
lies in the space $H^1(\Omega)$ (although it has a singularity in one point).

2. Let $\Omega = B(0, R) \subset \mathbb{R}^3$. In 3D, H^1 -functions can have singularities both at isolated points and along one-dimensional curves. Find or construct a function $g = g(x_1, x_2, x_3) \in H^1(\Omega)$, which has a singularity along 1D curve.

Hint: You can find an inspiration in 1.

(6 Points)

Exercise 3 Local P_k -Basis



The local P_k -basis on a d -dimensional simplex (triangle in 2D or tetrahedron in 3D) can be described by polynomials of a maximal degree k . In this exercise, we will restrict ourselves to a 2D reference element $\hat{\Omega}$.

As usual the source code can be found in the directory `uebungen/uebung05/` of the `dune-mpde` module. It will be shown (similar to the module `dune-localfunctions`), how the implementation of a local basis can be used to evaluate the function values.

1. Finish the implementation of `PkFunctor`, which should evaluate a function

$$f(x) = \sum_{i=1}^{n_k} \alpha_i \psi_i(x),$$

where the functions $(\psi_i)_{i \leq n_k}$, $\psi_i \in \mathbb{P}^k(\hat{\Omega})$ form a P_k -basis and $(\alpha_i)_{i \leq n_k} \in \mathbb{R}$ are the corresponding coefficients of the linear combination.

2. Once the functor is implemented, look at the output in Paraview. Describe qualitatively the characteristic properties of the P_k basis functions.
3. Implement the `MonomialFunctor` which should provide the same functionality for a monomial basis. You can reuse most of the `PkFunctor`.
4. Again, look at the output and describe the properties of the Monomial basis.
5. Show numerically that the P_k -functions really form a basis of the polynomials with maximal degree k on the reference element by constructing the transformation matrix from Monomial to P_k basis and checking its invertibility.

(10 Points)