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Exercise 1 *Domain regularity in 2D*

1. Decide if the following domains Ω are *Lipschitz domains*:

   (a)
   $$r > 1, \quad \Omega = \left\{(x,y) \in \mathbb{R}^2 | 0 < x < 1,\, |y| < x^r\right\}$$

   (b)
   $$\Omega_1 = \left\{(r,\theta) \in \mathbb{R}^2 | 0 < r < 1,\, 0 < \theta < \frac{3}{2}\pi\right\}$$
   $$\Omega_2 = \left\{(x,y) \in \mathbb{R}^2 | -0.5 < x < 0.5,\, y \geq |x|,\, y \leq 0.5\right\}$$
   $$\Omega = \Omega_1 \setminus \Omega_2$$

2. Find a domain in 2D that satisfies a *cone condition* but is not *Lipschitz*.

   Hint: For simply-connected domains, the Lipschitz property is equivalent to the cone condition.

(2 Points)

Exercise 2 *Robin boundary conditions*

Another frequently used type of boundary conditions involves a combination of function values and normal derivatives. Consider the model equation

$$\begin{aligned} -\nabla \cdot (a_1 \nabla u) + a_0 u &= f \quad \text{in} \quad \Omega, \\ u + \partial_n u &= g \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where $a_0 = 1, a_1 > 0$, $f \in C(\Omega)$ and $g \in C(\partial\Omega)$. Show, that the solution u of (1) fulfils the weak formulation

$$\int_\Omega (a_1 \nabla u \cdot \nabla v + a_0 u v) + \int_{\partial\Omega} a_1 u v = \int_\Omega f v + \int_{\partial\Omega} a_1 g v \qquad \forall v \in \mathcal{H}^1(\Omega).$$

Show for $f \in \mathcal{H}^1(\Omega)$ that the weak formulation has a unique solution $u \in \mathcal{H}^1(\Omega)$.

Hint: Follow the well-posedness proofs in the script.

Bonus: Prove the uniquenes of solution for the case when $a_0 = 0$.

(6 Points)

Exercise 3 *Approximation error*

Let $a : \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega) \to \mathbb{R}$ be a bilinearform $a(u, v) := (\nabla u, \nabla v)$ and $l : \mathcal{H}^1(\Omega) \to \mathbb{R}$ be a linear functional. In addition $V_h \subset \mathcal{H}_0^1(\Omega)$ be a finite-dimensional subspace and $u \in \mathcal{H}_0^1(\Omega)$, $u_h \in V_h$ fulfilling

$$a(u, v) = l(v), \quad \forall v \in \mathcal{H}_0^1(\Omega)$$

and

$$a(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h.$$

Show, that

$$\|\nabla u - \nabla u_h\|_0^2 = \|\nabla u\|_0^2 - \|\nabla u_h\|_0^2.$$

(3 Points)

Exercise 4 *Interpolation*

In this exercise you should investigate the property and convergence of interpolation using P_k basis functions. The programm in the directory *uebungen/uebung06* of the actual *dune-npde* modul interpolates a function

$$f(x) = \sum_{i=0}^{d} \frac{1}{x_i + 0.5}$$

in one and two dimensions to P_k space. The interpolation in 1D case is done on an interval $[0, 1]$ and in 2D on an unit triangle as in the previous exercise sheet.

The programm creates VTK files to visualize the reference function f, the interpolated function and the basis functions.

1. Have a look at programm and its structure.
   What happens in the function `interpolate_function()`?
2. The function `uniform_integration()` was changed (in comparison to last exercise). Describe the changes in the function `uniform_integration()` and give the reason for this necessity.
3. In `interpolate_function`, the interpolation result is wrapped in a `GridLevelFunction` before it is passed into the interpolation error quadrature. Why is this necessary? What does `GridLevelFunction` do? (Consider different grid levels being used across the program)
4. The programm computes the L_2 error of the interpolation. Is your observation consistent with your expectation? You can set parameters in *uebung06.ini*. Estimate (based on program output) the precision of the L_2 error on level 4 with $k = 4$.
5. Extend the program in 2D by using a unit square domain and a structured grid. Then switch to the Qk basis functions by using *Dune::PDELab::QkLocalFiniteElementMap*. Compare the L_2 error of P_1, P_2, Q_1 and Q_2 elements dependent on the number of degrees of freedom. Do you see any differences?
   Implement an alternative function

   $$g(x) = \begin{cases} 1 & \|x\| < 0.25 \\ 0 & \text{else} \end{cases}.$$

   Plot figures (L_2 error/number of degrees of freedom) of interpolation of f and g using polynomials of degree $1 \le k \le 4$ and explain the difference.

(10 Points)