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Exercise 1 *Homogeneous Dirichlet problem with $\mathbb{P}^1$ elements*

Let $\Omega = [a, b] \subset \mathbb{R}$ be a real 1D domain and $\mathcal{T}_N$ be a equidistant grid on Ω with grid size $h = (b-a)/N$ for $N \in \mathbb{N}$. Let

$$V = \{v \in H^1(\Omega) \mid v(a) = v(b) = 0\}$$

be a vector space and

$$V_h = \{v_h \in \mathbb{C}^0(\Omega) \mid \forall s \in \mathcal{T} : v_h\big|_s \in \mathbb{P}^1(s) \quad \wedge \quad v_h(a) = v_h(b) = 0\}$$

be a finite-dimensional subspace. In addition let l be a continuous linear form $l : V \to \mathbb{R}$ and define a bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx.$$

The vectors $u \in V$ and $u_h \in V_h$ fulfill

$$a(u, v) = l(v), \qquad \forall v \in V$$

and

$$a(u_h, v_h) = l(v_h), \qquad \forall v_h \in V_h.$$

1. Show, that $(\cdot,\cdot)_V = a(\cdot,\cdot)$ induces a scalar product on V.
2. Show, that $u(a + ih) = u_h(a + ih)$ for $i \in 0, \dots, N$.

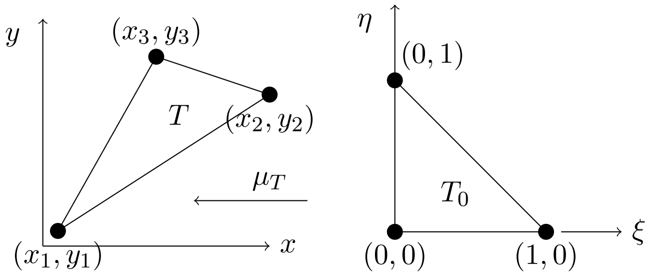
   Hints: Choose a simple basis for the test space; You will derive a system of equations with a unique solution.

(7 Points)

Exercise 2 *Local $\mathbb{P}^1$ basis on reference element*

Let $d = 2$. We consider a unit triangle T_0 with nodes $n_0 = (0, 0)$, $n_1 = (1, 0)$, $n_2 = (0, 1)$ and an arbitrary triangle $T \subset \mathbb{R}^2$ with nodes $a_0 = (x_0, y_0)$, $a_1 = (x_1, y_1)$, $a_2 = (x_2, y_2)$, see picture.

1. The linear function $p \in \mathbb{P}_1(T_0)$ on T_0 can be defined using values in points n_i and the definition is unique. Find a node-basis $\tilde{\varphi}_i,\ i = 1, 2, 3$ of $\mathbb{P}_1(T_0)$ fulfilling $\tilde{\varphi}_i(n_j) = \delta_{ij}$.



2. Find a reference affine mapping $\mu_T : T_0 \to T$. Is this mapping unique and invertible?
3. The functions $\varphi_i,\ i = 1, 2, 3$ are given by

$$\varphi_i(\xi, \eta) := \tilde{\varphi}_i(\mu_T^{-1}(\xi, \eta)).$$

   Prove, that $\varphi \in \mathbb{P}_1(T_0)$ and $\varphi_i(a_j) = \delta_{i,j}$.

4. If you want to integrate a function $v \in \mathbb{P}_1(T)$ on the T, you can first integrate it on the reference element T_0 (no change in quadrature points) and the result should be modified (regarding original element). Which factor should stay in front of the second integral?

$$\int_T v(x,y)dxdy = \cdots \int_{T_0} v\left(\mu_T\left(\xi,\eta\right)\right) d\xi d\eta.$$

(5 Points)

Exercise 3 *Elliptic operator in PDELab*

In this exercise you will solve a PDE numerically for the first time. The program in the directory *uebungen/uebung08* in *dune-npde* solves a generic convection-diffusion problem

$$\begin{aligned}
\nabla \cdot (-A(x)\nabla u + b(x)u) + c(x)u &= f \text{ in } \Omega, \\
u &= g \text{ on } \partial\Omega_D(Dirichlet) \\
(b(x,u) - A(x)\nabla u) \cdot n &= j \text{ on } \partial\Omega_N(Neumann) \\
-(A(x)\nabla u) \cdot n &= o \text{ on } \partial\Omega_O(Outflow)
\end{aligned}$$

using P^k finite elements on a square domain. The specific parameters are defined in problem.hh.

We do not need to implement the actual bilinear form ourselves, as that is already provided by PDELab via *ConvectionDiffusionFEM*.

1. Try to understand what the code does. What do the solver parameters mentioned in `uebung08.cc` (see *TODO* tag) do?
2. Write down explicitly the PDE being solved as well as the analytical solution. (It is implemented in the Dirichlet parameter, which is also used as the reference solution in the error computation.)
3. Look up *SubsamplingVTKWriter* in the DUNE documentation and use it instead of the *VTKWriter*. Note that it requires an additional integer argument for its constructor. Compare VTK outputs of the subsampling and non-subsampling versions. What does SubsamplingVTKWriter do?
4. You can set different polynomial degrees in `uebung08.ini`. What L_2 convergence rates do you expect? Do the numerical results match your expectation?
5. As you can see, the error computation in H_1 seminorm is not implemented yet. First, implement the gradient of the solution in `exact_gradient.hh`, we need it as our analytical reference. Then, implement the H_1 seminorm error computation in analogy to the L_2 version. What convergence rates do you get? What convergence rate would you expect for the full H_1 norm?
6. Play around with `problem.hh`. For example, set Neumann conditions on parts of the boundary. Why do convergence rate computations break?

(10 Points)