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Exercise 1 *Tricomi Equation*

Consider the Tricomi equation

$$\partial_y^2 u + y\partial_x^2 u = 0$$

for a scalar-valued function u on the domain

$$\Omega = [-1,1] \times [-1,1].$$

Determine which class of equation it is.

This nonlinear equation can be used to model an object travelling at supersonic speed.

(2 Points)

Exercise 2 *Weak differentiability*

Continuous, piecewise-smooth functions in 1D are weakly differentiable (see example 5.31 in the lecture notes). The continuity of the function is crucial.

Consider the function

$$f(x) = \begin{cases} -1 & x \in (-1,0] \\ 1 & x \in (0,1) \end{cases}$$

and show that the weak derivative of f does not exist. **(2 Points)**

Exercise 3 *Projections*

Let Y be a subspace of a normed vector space X. An operator $P : X \to X$ is said to be a projection on Y if

$$P^2 = P \quad \text{and} \quad \text{Range}(P) = Y.$$

Show the following:

1. P is a projection if and only if $P : X \to Y$ and $P = I$ on Y.
2. If P is a projection, then $X = \text{Ker}(P) \oplus \text{Range}(P)$, where $\oplus$ denotes a direct sum.

(4 Points)

Exercise 4 *Operators on Hilbert space*

Let H be a Hilbert space and Y a closed subspace of H. Define the map $P : H \to Y$ for each $v \in H$ as

$$\forall y \in Y : (P(v), y) = (v, y).$$

Let us prove that:

1. Operator P is linear and continuous.
2. For $v \in H$ it holds
$$\|P(v) - v\| = \min_{y \in Y} \|y - v\|.$$

Hint: Apply *Lax-Milgram Theorem* and *Characterization Theorem*.

(5 Points)

Exercise 5 *Riesz Theorem (constructive proof)*

1. Find a constructive proof of Riesz Theorem:

   Let $(V, (.,.)_V)$ be a real Hilbert space and $v' \in V'$ an arbitrary linear form on V. Then there exists a unique $u \in V$ such that

   $$\langle v', w\rangle_{V',V} = (u, w)_V \qquad \forall w \in V.$$

   Moreover, $\|v'\|_{V'} = \|u\|_V$.

   Hints:

   (a) First prove the uniqueness (under the assumption of an existence) of u.

   (b) Let $M = \{w \in V \,|\, \langle v', w\rangle_{V',V} = 0\}$. Show that $M^\perp$ is a one-dimensional subspace of V (or $v' = 0$ holds) and that $V = M \oplus M^\perp$ holds.

   (c) Show that for $z \in M^\perp$ the vector u is given by

   $$u = \frac{\langle v', z\rangle_{V',V}}{\|z\|_V^2} z.$$

2. After proving Riesz Theorem, show the second part:
   The map $\tau : V' \to V$ mapping $v' \in V'$ to the corresponding $u \in V$ is linear and an isometry, i.e. $\|\tau v'\|_V = \|v'\|_{V'}$.

(6 Points)