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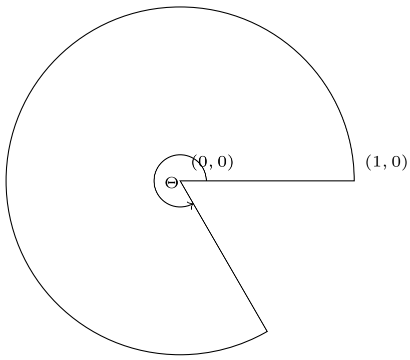
Exercise 1 *H^1 functions*

Let $\Omega \subset \mathbb{R}^2$ be the unit cube, $\Omega = [0,1] \times [0,1]$.

1. For which α is the function in polar coordinates
$$f(r, \varphi) = r^\alpha \, sin(\alpha\varphi) \tag{1}$$
from space $H^1(\Omega)$?

2. The function (1) is a special form of the harmonic function
$$g(r, \varphi) = r^{\frac{\pi}{\Theta}} \sin(\frac{\pi}{\Theta}\varphi).$$
Show that g is harmonic, that means $\Delta g = 0$.

   The Laplace-Problem $\Delta u = 0$ should be solved on a domain Ω (see figure). What Dirichlet conditions must be set such that g is a solution? Write them down explicitly.



(4 Points)

Exercise 2 *Discontinuity of H^1-functions in 2D and 3D*

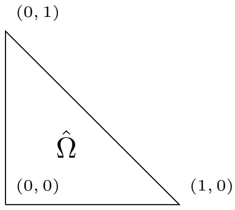
1. Consider the domain $\Omega = B(0, R) \subset \mathbb{R}^2$, where
$$B(0, R) = \{x \in \mathbb{R}^2 \mid \|x\| < R\}, \quad 0 < R < \frac{1}{e}.$$
Show in detail that the function
$$f(x) = \ln\left(\ln\left(\frac{1}{r(x)}\right)\right), \quad r(x) = \left(\sum_{i=1}^{2} x_i^2\right)^{\frac{1}{2}}$$
lies in the space $H^1(\Omega)$ (although it has a singularity in one point).

2. Let $\Omega = B(0, R) \subset \mathbb{R}^3$. In 3D, H^1-functions can have singularities both at isolated points and along one-dimensional curves. Find or construct a function $g = g(x_1, x_2, x_3) \in H^1(\Omega)$, which has a singularity along 1D curve.

   Hint: You can find an inspiration in 1.

(6 Points)

Exercise 3 *Local Pk-Basis*



The local Pk-basis on a d-dimensional simplex (triangle in 2D or tetrahedron in 3D) can be described by polynomials of a maximal degree k. In this exercise, we will restrict ourselves to a 2D reference element $\hat{\Omega}$.

As usual the source code can be found in the directory *uebungen/uebung05/* of the *dune-npde* module. It will be shown (similar to the module *dune-localfunctions*), how the implementation of a local basis can be used to evaluate the function values.

1. Finish the implementation of *PkFunctor*, which should evaluate a function

$$f(x) = \sum_{i=1}^{n_k} \alpha_i \psi_i(x),$$

   where the functions $(\psi_i)_{i \leq n_k}, \psi_i \in \mathbb{P}^k(\hat{\Omega})$ form a Pk-basis and $(\alpha_i)_{i \leq n_k} \in \mathbb{R}$ are the corresponding coefficients of the linear combination.

2. Once the functor is implemented, look at the output in Paraview. Describe qualitatively the characteristic properties of the Pk basis functions.
3. Implement the *MonomialFunctor* which should provide the same functionality for a monomial basis. You can reuse most of the *PkFunctor*.
4. Again, look at the output and describe the properties of the Monomial basis.
5. Show numerically that the Pk-functions really form a basis of the polynomials with maximal degree k on the reference element by constructing the transformation matrix from Monomial to Pk basis and checking its invertibility.

(10 Points)