
Exercise 1 *Domain regularity in 2D*

1. Decide if the following domains Ω are *Lipschitz domains*:

   (a)
   $$r > 1, \quad \Omega = \left\{(x,y) \in \mathbb{R}^2 | 0 < x < 1,\, |y| < x^r\right\}$$

   (b)
   $$\begin{aligned} \Omega_1 &= \left\{(r,\theta) \in \mathbb{R}^2 | 0 < r < 1,\, 0 < \theta < \frac{3}{2}\pi\right\} \\ \Omega_2 &= \left\{(x,y) \in \mathbb{R}^2 | -0.5 < x < 0.5,\, y \geq |x|,\, y \leq 0.5\right\} \\ \Omega &= \Omega_1 \setminus \Omega_2 \end{aligned}$$

2. Find a domain in 2D that satisfies a *cone condition* but is not *Lipschitz*.

   Hint: For simply-connected domains, the Lipschitz property is equivalent to the cone condition.

(2 Points)

Exercise 2 *Robin boundary conditions*

Another frequently used type of boundary conditions involves a combination of function values and normal derivatives. Consider the model equation

$$\begin{aligned} -\nabla \cdot (a_1 \nabla u) + a_0 u &= f \quad \text{in} \quad \Omega, \\ u + \partial_n u &= g \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where $a_0 = 1, a_1 > 0$, $f \in C(\Omega)$, $g \in C(\partial\Omega)$, and n is the unit outer normal vector. Show, that the solution u of (1) fulfills the weak formulation

$$\int_\Omega (a_1 \nabla u \cdot \nabla v + a_0 uv) + \int_{\partial\Omega} a_1 uv = \int_\Omega fv + \int_{\partial\Omega} a_1 gv \qquad \forall v \in \mathcal{H}^1(\Omega).$$

Show for $f \in \mathcal{H}^1(\Omega)$ that the weak formulation has a unique solution $u \in \mathcal{H}^1(\Omega)$.

Hint: Follow the well-posedness proofs in the script.

Bonus: Prove the uniqueness of the solution for the case when $a_0 = 0$.

(6 Points)

Exercise 3 *Approximation error*

Let $a : \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega) \to \mathbb{R}$ be a bilinear form $a(u, v) = (\nabla u, \nabla v)$, and $l : \mathcal{H}^1(\Omega) \to \mathbb{R}$ be a linear functional. In addition, $V_h \subset \mathcal{H}^1_0(\Omega)$ be a finite-dimensional subspace, and $u \in \mathcal{H}^1_0(\Omega)$, $u_h \in V_h$ fulfill

$$a(u, v) = l(v), \quad \forall v \in \mathcal{H}^1_0(\Omega)$$

and

$$a(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h.$$

Show that

$$\|\nabla u - \nabla u_h\|_0^2 = \|\nabla u\|_0^2 - \|\nabla u_h\|_0^2.$$

(3 Points)

Exercise 4 *Interpolation*

In this exercise, you should investigate the property and convergence of interpolation using P_k basis functions. The program in the directory *uebungen/uebung06* of the module *dune-npde* interpolates a function

$$f(x) = \sum_{i=0}^{d} \frac{1}{x_i + 0.5}$$

in one and two dimensions to P_k space. The interpolation in 1D case is done on an interval $[0, 1]$ and in 2D on a unit triangle as in the previous exercise sheet.

The program creates VTK files to visualize the reference function f, the interpolated function, and basis functions.

1. Have a look at program and its structure.
   What happens in the function `interpolate_function()`?
2. The function `uniform_integration()` was changed (in comparison to the last exercise). Describe the changes in the function `uniform_integration()` and give the reason for this necessity.
3. In `interpolate_function`, the interpolation result is wrapped in a `GridLevelFunction` before it is passed into the interpolation error quadrature. Why is this necessary? What does `GridLevelFunction` do? (Consider different grid levels being used across the program.)
4. The program computes the L_2 error of the interpolation. Is your observation consistent with your expectation? You can set parameters in *uebung06.ini*. Estimate (based on program output) the precision of the L_2 error on level 4 with $k = 4$.
5. Extend the program to 2D by using a unit square domain and a structured grid. Then switch to the Qk basis functions by using *Dune::PDELab::QkLocalFiniteElementMap*. Compare the L_2 error of P_1, P_2, Q_1, and Q_2 elements depending on the number of degrees of freedom. Do you see any differences?
   Implement an alternative function

   $$g(x) = \begin{cases} 1 & \|x\| < 0.25 \\ 0 & \text{else} \end{cases}.$$

   Plot figures (L_2 error/number of degrees of freedom) of interpolation of f and g using polynomials of degree $1 \le k \le 4$ and explain the difference.

(10 Points)