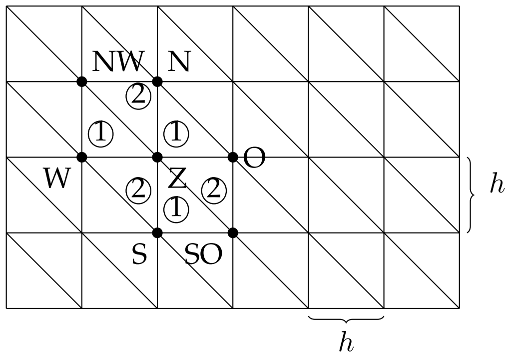

Exercise 1 *Stiffness Matrix*

We want to solve homogeneous Laplace equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

with P_1 elements on the following grid:



Basis functions of all inner nodes look the same and therefore all rows of the stiffness matrix are identical (except for boundary nodes). As a consequence, it is sufficient to look at only one node Z. Let N, O, SO, S, W, NW denote the neighbours of Z.

Determine the matrix values of one row of the stiffness matrix corresponding to the inner node. In order to do that you have to choose a numeration of basis functions. Express your solution as *finite difference stencil* analogue to finite difference methods.

(5 Points)

Exercise 2 *Bramble-Hilbert in 1D*

Let $\Omega = [a, b] \subset \mathbb{R}$, $w : \Omega \to \mathbb{R}$ be a function with $w \in H^2(\Omega)$. Let x_k be the vertices of a triangulation of Ω with $x_k = a + \sum_{i=1}^{k} h_i$, $k = 0 \dots N$ and $h_k > 0$ such that $x_0 = a$ and $x_N = b$ holds. Let v be a piecewise linear interpolation of w fulfilling $v(x_i) = w(x_i)$ for $i = 0 \dots N$. Let $\hat{\Omega} = [0, 1]$ be the reference element and $\mu_k : \hat{\Omega} \to [x_{k-1}, x_k]$ be the corresponding transformation to grid cell $[x_{k-1}, x_k]$.

Show that for $e(x) := w - v$ and $\hat{e}_k(\hat{x}) := e(\mu_k(\hat{x}))$ it holds

$$|\hat{e}_k|_{1,\hat{\Omega}} \le \|\partial_{\hat{x}\hat{x}} \hat{e}_k\|_{0,\hat{\Omega}}$$

and with $h = \max_{1 \le k \le N} \{h_k\}$ it holds

$$\|e\|_{1,\Omega}^2 \le h^2(h+1)\|\partial_{xx} w\|_{0,\Omega}^2.$$

(5 Points)

Exercise 3 *Convergence Rates for Poisson Equation*

Let $\Omega = [0,a] \times [0,b] \subset \mathbb{R}^2,\ 0 < a, b \in \mathbb{R}$. The Poisson equation

$$\begin{aligned} -\Delta u(x,y) &= \left(\frac{3b}{2}y^2 - \frac{b^2}{2}y - y^3\right)(6x - 3a) \\ &\quad + \left(\frac{3a}{2}x^2 - \frac{a^2}{2}x - x^3\right)(6y - 3b), \quad (x,y) \in \Omega \end{aligned} \tag{1}$$

with homogeneous Dirichlet boundary condition has the analytical solution

$$u(x,y) = xy(a-x)(b-y)\left(\frac{a}{2} - x\right)\left(\frac{b}{2} - y\right).$$

In *uebungen/uebung09* of your *dune-npde* module you can find a program that solves a Poisson equation (1) with P^k finite element on a conform triangular grid (*UGGrid*) and with Q^k finite element on a conforming quadrilateral grid (*YaspGrid*). For the domain we choose $\Omega = [0,2] \times [0,2] \subset \mathbb{R}^2$.

1. The program can determine norms $\|u - u_h\|_{0,\Omega}$, $\|\nabla(u - u_h)\|_{0,\Omega}$, and $\|u - u_h\|_{L_\infty(\Omega)}$. Extend it such that it also calculates $\|u - u_h\|_{1,\Omega}$ and prints the corresponding convergence rate.
2. Plot $\|u - u_h\|_{0,\Omega}$, $\|u - u_h\|_{1,\Omega}$, and $\|u - u_h\|_{L_\infty(\Omega)}$ against the number of degrees of freedom for P^k and Q^k elements, $k = 1, 2$. Use logarithmic scale on both axis. How can you visually see the convergence rate in such a plot?
3. Implement the function `gridFunctionMax` such that it calculates

   $$f(u_h, \Omega) = \max_i |u(a_i) - u_h(a_i)|, \text{ where } a_0, \ldots, a_{N-1} \in \overline{\Omega} \text{ are the vertices of our grid.}$$

   What results do you get for P^1 elements?

(10 Points)