Exercises for the Lecture Series

"Object-Oriented Programming for Scientific Computing"

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Exercise 1 Rational Numbers

Write a class for rational numbers. The number should always be represented as a *fully reduced fraction* of the form

$$\frac{\text{numerator}}{\text{denominator}}$$

with denominator > 0.

1. What is an appropriate data structure for rational numbers?
2. Start by writing a function `int gcd(int,int)` (greatest common divisor), you will need it to reduce fractions.
   - You can use the Euclidean algorithm to determine the greatest common divisor.
   - For an algorithm see
     `http://en.wikipedia.org/wiki/Greatest_common_divisor`
   - Implement this scheme as a recursive function.
3. Write a class `Rational`, which represents a rational number. The constructor should have the numerator and the denominator as arguments. Be sure to check for valid input. In addition, the class has two functions `numerator()` and `denominator()` that return the values of the numerator and denominator. The class should have three constructors:
   - a default constructor that initializes the fraction with 1,
   - a constructor that initializes the fraction with a given numerator and denominator and
   - a constructor that initializes the fraction with a given whole number.
4. Supplement the class with operators for `*=` `+=` `-=` `/=` and `==`.
5. Use the newly implemented methods to implement free operators `*` `+` `-` `/`.
6. Check your implementation using various test cases. Initialize three fractions

$$f_1 = -\frac{3}{12}, \quad f_2 = \frac{4}{3}, \quad f_3 = \frac{0}{1}.$$

Test the operators with the following examples:

$$f_3 = f_1 + f_2, \quad f_3 = f_1 \cdot f_2, \quad f_3 = 4 + f_2, \quad f_3 = f_2 + 5, \quad f_3 = 12 \cdot f_1, \quad f_3 = f_1 \cdot 6, \quad f_3 = \frac{f_1}{f_2}.$$

Print the result after each operation. The corresponding solutions are:

$$\frac{13}{12}, \quad -\frac{1}{3}, \quad \frac{16}{3}, \quad \frac{19}{3}, \quad -\frac{3}{1}, \quad -\frac{3}{2}, \quad -\frac{3}{16}.$$

10 Points

Exercise 2 Farey Sequences

A Farey sequence F_N of degree N (Farey fractions of degree N) is an ordered set of reduced fractions

$$\frac{p_i}{q_i} \quad \text{with} \quad p_i \le q_i \le N \quad \text{and} \quad 0 \le i < |F_N|$$

and

$$\frac{p_i}{q_i} < \frac{p_j}{q_j} \qquad \forall\, 0 \le i < j < |F_N|.$$

Use the class `Rational` from the first exercise to write a function

```
void Farey(int N)
```

which calculates the Farey fractions up to degree N and prints the resulting Farey sequences up to degree N on the screen.

Algorithm: The sequences can be computed recursively. The first sequence is given by

$$F_1 = \left(\tfrac{0}{1}, \tfrac{1}{1}\right)$$

For a known sequence F_N one can get F_{N+1} by inserting an additional fraction $\frac{p_i+p_{i+1}}{q_i+q_{i+1}}$ between two consecutive entries $\frac{p_i}{q_i}$ and $\frac{p_{i+1}}{q_{i+1}}$ if $q_i + q_{i+1} = N + 1$ holds for the sum of denominators.

Example: Determining F_7 from F_6 results in the following construction:

$$F_6 = \left(\underbrace{\tfrac{0}{1}, \tfrac{1}{6}}_{\frac{1}{7}}, \tfrac{1}{5}, \underbrace{\tfrac{1}{4}, \tfrac{1}{3}}_{\frac{2}{7}}, \underbrace{\tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}}_{\frac{3}{7} \text{ and } \frac{4}{7}}, \underbrace{\tfrac{2}{3}, \tfrac{3}{4}}_{\frac{5}{7}}, \tfrac{4}{5}, \underbrace{\tfrac{5}{6}, \tfrac{1}{1}}_{\frac{6}{7}} \right)$$

The new elements are:

$$\tfrac{0+1}{1+6} = \tfrac{1}{7} \;;\; \tfrac{1+1}{4+3} = \tfrac{2}{7} \;;\; \tfrac{2+1}{5+2} = \tfrac{3}{7} \;;\; \tfrac{1+3}{2+5} = \tfrac{4}{7} \;;\; \tfrac{2+3}{3+4} = \tfrac{5}{7} \;;\; \tfrac{5+1}{6+1} = \tfrac{6}{7}$$

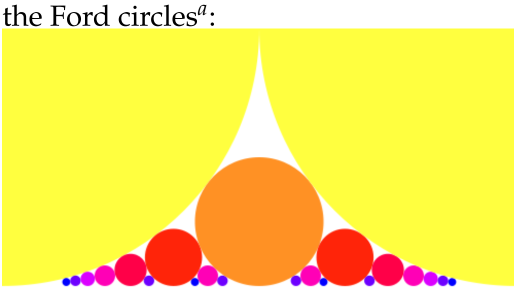
The sorted sequence then is:

$$F_7 = \left(\tfrac{0}{1}, \mathbf{\tfrac{1}{7}}, \tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{4}, \mathbf{\tfrac{2}{7}}, \tfrac{1}{3}, \tfrac{2}{5}, \mathbf{\tfrac{3}{7}}, \tfrac{1}{2}, \mathbf{\tfrac{4}{7}}, \tfrac{3}{5}, \tfrac{2}{3}, \mathbf{\tfrac{5}{7}}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \mathbf{\tfrac{6}{7}}, \tfrac{1}{1} \right)$$

For checking:
The Farey sequences up to degree 6

$$\begin{aligned}
F_1 &= \left(\tfrac{0}{1}, \tfrac{1}{1}\right) \\
F_2 &= \left(\tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{1}\right) \\
F_3 &= \left(\tfrac{0}{1}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{1}\right) \\
F_4 &= \left(\tfrac{0}{1}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{1}{1}\right) \\
F_5 &= \left(\tfrac{0}{1}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{1}{1}\right) \\
F_6 &= \left(\tfrac{0}{1}, \tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \tfrac{1}{1}\right).
\end{aligned}$$

There is a beautiful illustration of these fractions, the Ford circles[a]:



[a] see `http://en.wikipedia.org/wiki/Ford_circle`

10 Points