Dr. Ole Klein, Stefan Meggendorfer
IWR, Heidelberg University
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Exercise Sheet 2

Exercise 1: C++ Quiz (5 points)

On https://cppquiz.org you can find a quiz with C++ specific questions. In this exercise, answer the following questions:

Question 1: https://cppquiz.org/quiz/question/197 (variable lifetime)

Question 2: https://cppquiz.org/quiz/question/161 (Duff's Device)

Question 3: https://cppquiz.org/quiz/question/9 (reference arguments)

Question 4: https://cppquiz.org/quiz/question/113 (overload resolution)

Question 5: https://cppquiz.org/quiz/question/5 (initialization order)

The questions are sorted (more or less) according to the structure of the lecture. For questions 1, 3, 4, and 5, write a short statement what information the question and its solution are trying to convey. Regarding question 2: inform yourself about the construct that is used. What is its purpose? Would you suggest using this in real-world code? Why, or why not?

Exercise 2: Rational Numbers (10 points)

Write a class for rational numbers. The number should always be represented as a *fully reduced fraction* of the form

$$\frac{\text{numerator}}{\text{denominator}}$$

with denominator > 0.

(a) What is an appropriate data structure for rational numbers?

(b) Start by writing a function `int gcd(int,int)` (greatest common divisor), you will need it to reduce fractions.

- You can use the Euclidean algorithm to determine the greatest common divisor.
- For an algorithm see https://en.wikipedia.org/wiki/Greatest_common_divisor
- Implement this scheme as a recursive function.

(c) Write a class `Rational`, which represents a rational number. The constructor should have the numerator and the denominator as arguments. Be sure to check for valid input. In addition, the class has two functions `numerator()` and `denominator()` that return the values of the numerator and denominator. The class should have three constructors:

- a default constructor that initializes the fraction with 1,
- a constructor that initializes the fraction with a given numerator and denominator, and
- a constructor that initializes the fraction with a given whole number.

(d) Supplement the class with operators for `*=` `+=` `-=` `/=` and `==`.

(e) Use the newly implemented methods to implement free operators `*` `+` `-` `/`.

(f) Check your implementation using various test cases. Initialize three fractions

$$f_1 = -\frac{3}{12}, \quad f_2 = \frac{4}{3}, \quad f_3 = \frac{0}{1}.$$

Test the operators with the following examples:

$$f_3 = f_1 + f_2, \quad f_3 = f_1 \cdot f_2, \quad f_3 = 4 + f_2, \quad f_3 = f_2 + 5,$$
$$f_3 = 12 \cdot f_1, \quad f_3 = f_1 \cdot 6, \quad f_3 = \frac{f_1}{f_2}.$$

Print the result after each operation. The corresponding solutions are:

$$\frac{13}{12}, \quad -\frac{1}{3}, \quad \frac{16}{3}, \quad \frac{19}{3}, \quad -\frac{3}{1}, \quad -\frac{3}{2}, \quad -\frac{3}{16}.$$

Exercise 3: Farey Sequences (10 points)

A Farey sequence F_N of degree N (or: the Farey fractions of degree N) is an ordered set of reduced fractions

$$\frac{p_i}{q_i} \qquad \text{with} \qquad p_i \leq q_i \leq N \qquad \text{and} \qquad 0 \leq i < |F_N|$$

and

$$\frac{p_i}{q_i} < \frac{p_j}{q_j} \qquad \forall\, 0 \leq i < j < |F_N|.$$

Use the class `Rational` from the previous exercise to write a function

```
void Farey(int N)
```

which calculates the Farey fractions up to degree N and prints the resulting Farey sequences up to degree N on the screen.

Algorithm: The sequences can be computed recursively. The first sequence is given by

$$F_1 = \left(\tfrac{0}{1}, \tfrac{1}{1}\right)$$

For a known sequence F_N one can get F_{N+1} by inserting an additional fraction $\frac{p_i+p_{i+1}}{q_i+q_{i+1}}$ between two consecutive entries $\frac{p_i}{q_i}$ and $\frac{p_{i+1}}{q_{i+1}}$ if $q_i + q_{i+1} = N + 1$ holds for the sum of denominators.

Example: Determining F_7 from F_6 results in the following construction:

$$F_6 = \Big(\underbrace{\tfrac{0}{1}, \tfrac{1}{6}}_{\frac{1}{7}}, \tfrac{1}{5}, \underbrace{\tfrac{1}{4}, \tfrac{1}{3}}_{\frac{2}{7}}, \underbrace{\tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}}_{\frac{3}{7} \text{ and } \frac{4}{7}}, \underbrace{\tfrac{2}{3}, \tfrac{3}{4}}_{\frac{5}{7}}, \tfrac{4}{5}, \underbrace{\tfrac{5}{6}, \tfrac{1}{1}}_{\frac{6}{7}}\Big)$$

The new elements are:

$$\tfrac{0+1}{1+6} = \tfrac{1}{7} \;\; ; \;\; \tfrac{1+1}{4+3} = \tfrac{2}{7} \;\; ; \;\; \tfrac{2+1}{5+2} = \tfrac{3}{7} \;\; ; \;\; \tfrac{1+3}{2+5} = \tfrac{4}{7} \;\; ; \;\; \tfrac{2+3}{3+4} = \tfrac{5}{7} \;\; ; \;\; \tfrac{5+1}{6+1} = \tfrac{6}{7}$$

The sorted sequence is then:

$$F_7 = \left(\tfrac{0}{1}, \boldsymbol{\tfrac{1}{7}}, \tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{4}, \boldsymbol{\tfrac{2}{7}}, \tfrac{1}{3}, \tfrac{2}{5}, \boldsymbol{\tfrac{3}{7}}, \tfrac{1}{2}, \boldsymbol{\tfrac{4}{7}}, \tfrac{3}{5}, \tfrac{2}{3}, \boldsymbol{\tfrac{5}{7}}\; \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \boldsymbol{\tfrac{6}{7}}, \tfrac{1}{1}\right)$$

For checking: The Farey sequences up to degree 6 are

$$F_1 = \left(\tfrac{0}{1}, \tfrac{1}{1}\right), \quad F_2 = \left(\tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{1}\right), \quad F_3 = \left(\tfrac{0}{1}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{1}\right), \quad F_4 = \left(\tfrac{0}{1}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{1}{1}\right),$$
$$F_5 = \left(\tfrac{0}{1}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{1}{1}\right), \quad F_6 = \left(\tfrac{0}{1}, \tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \tfrac{1}{1}\right).$$

There is a beautiful illustration of these fractions, the Ford circles[1].

[1] see https://en.wikipedia.org/wiki/Ford_circle