
Exercise 1 *Jacobi iteration as additive Schwarz*

Let A be the stiffness matrix of a Finite Element discretization with the Finite Element space V_h and basis φ_i^h. We will use the unique representation

$$u_h = \sum_{i=1}^{N_h} x_i \varphi_i^h, \quad x_i \in \mathbb{R}$$

of any function $u_h \in V_h$ as a one dimensional, non-overlapping decomposition of the index sets, i.e. $V_{h,i} = \text{span}\{\varphi_i^h\}$ (cf. exercise 9).

In the lecture you have learned the additive Schwarz iteration

$$x^{(k+1)} = x^{(k)} + \omega \sum_{i=1}^{p} R_i^T A_i^{-1} R_i (b - Ax^{(k)}) \quad \text{with} \quad A_i = R_i A R_i^T. \tag{1}$$

1. Specify how the damping factor ω, the number of subdomains p and the restriction matrices R_i need to be chosen such that (1) describes the Jacobi iteration.

Assume that the following estimate holds:

$$\|x\|_2 \leq C h^{-\frac{d}{2}} \|u_h\|_{0,\Omega}$$

where $\|\cdot\|_2$ denotes the Euclidian norm on $\mathbb{R}^{N_h}$ and C is a constant independent on h.

In order to apply the abstract Schwarz theory to the Jacobi iteration, the following two assumptions need to be fulfilled:
Assumption 1 (Stable splitting)**.**
There exists a constant C_0 such that for all $x \in \mathbb{R}^{N_h}$ there exists a splitting $x = \sum_{i=1}^{p} R_i^T x_i$ such that

$$\sum_{i=1}^{p} \langle R_i^T x_i, R_i^T x_i \rangle_A \leq C_0 \langle x, x \rangle_A.$$

Assumption 2 (Strengthened Cauchy-Schwarz inequality)**.**
There exist constants $0 \leq \varepsilon_{i,j} \leq 1$ for $1 \leq i, j \leq p$ such that for all x_i and x_j it holds

$$|\langle R_i^T x_i, R_j^T x_j \rangle_A| \leq \varepsilon_{ij} \langle R_i^T x_i, R_i^T x_i \rangle_A^{\frac{1}{2}} \langle R_j^T x_j, R_j^T x_j \rangle_A^{\frac{1}{2}}.$$

2. Show that these assumptions are satisfied by the Jacobi iteration and specify the best possible choice of the constants C_0 and ε_{ij}.

(8 Points)

Exercise 2 *Optimal upper bound for the additive Schwarz method*

Purpose of this excercise is to show a better bound for the largest eigenvalue for the additive overlapping Schwarz method than is given by Lemma 4.5 and Remark 4.6 in the lecture notes.

It takes the form

$$\langle \sum_{i=0}^{p} P_i x, x \rangle_A \leq (1 + n_c) \langle x, x \rangle_A \tag{2}$$

where n_c is the number of colors in the parallelized multiplicative Schwarz method introduced in Observation 3.4. In order to prove this result proceed as follows:

1. With the notation of Observation 3.4 define

   $$\mathbb{P}_n = \sum_{i \in J_n} P_i, \qquad \text{i.e.} \qquad \sum_{i=1}^{p} P_i = \sum_{n=1}^{n_c} \sum_{i \in J_n} P_i = \sum_{n=1}^{n_c} \mathbb{P}_n.$$

   Note that the coarse grid (index $i = 0$) is excluded.

2. Now show that the $\mathbb{P}_n$ are A-orthogonal projections, i.e. they satisfy the same properties as the P_i in Lemma 4.4.

3. Then estimate $\langle \sum_{n=1}^{n_c} \mathbb{P}_n x, x \rangle_A$ using the properties of $\mathbb{P}_n$ and the Cauchy-Schwarz inequality.

(8 Points)