Prof. Dr. Peter Bastian, Linus Seelinger Submission date: 28 June 2018

IWR, Universität Heidelberg

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**Exercise 1** *Coarse grid correction*

In the lecture you have learned the coarse grid correction. In this exercise we are going to derive the algebraic formulation from the variational formulation.
Let $\mathcal{T}_H$ be a triangulation of the domain $\Omega$ and let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$. The corresponding the Finite Element bases are denoted by

$$\Phi_H = \{\varphi_i^H \mid i \in \mathcal{I}_H\},$$
$$\Phi_h = \{\varphi_i^h \mid i \in \mathcal{I}_h\}$$

and the Finite Element spaces by

$$V_H = \operatorname{span} \Phi_H,$$
$$V_h = \operatorname{span} \Phi_h.$$

Define the restriction $R_H : V_h \to V_H$ which is represented in the bases $\Phi_H$, $\Phi_h$ above by the matrix

$$(R_H)_{ij} = \varphi_i^H(x_j), \quad i \in \mathcal{I}_H, \quad j \in \mathcal{I}_h$$

where the $x_j$ are the Lagrange nodal points: $\varphi_i^h(x_j) = \delta_{ij}$.

Let $u_h^{(k)}$ be given. The coarse grid correction $w_H$ is computed with the variational formulation

$$a(u_h^{(k)} + w_H, v) = l(v) \qquad \forall v \in V_H.$$

1. Show that the following equation holds:

$$\varphi_i^H = \sum_{j \in \mathcal{I}_h} (R_H)_{ij} \varphi_j^h.$$

2. Show the relation

$$R_H A_h (R_H)^T = A_H,$$

with

$$A_h = a(\varphi_j^h, \varphi_i^h),$$
$$A_H = a(\varphi_j^H, \varphi_i^H).$$

3. Derive from the variational formulation the algebraic version presented in the lecture.

**( 6 Points )**

**Exercise 2** *Localized Stable Splitting*

We consider the two level Schwarz method with coarse grid correction based on the hierarchical construction where $\mathcal{T}_H$ is a coarse mesh of $p$ nonoverlapping subdomains which is uniformly refined to give a fine mesh $\mathcal{T}_h$. Then overlapping subdomains $\hat{\Omega}_j$ are formed by adding elements $t \in \mathcal{T}_h$ from neighboring subdomains. Thus $\mathcal{T}_{h,j} = \{t \in \mathcal{T}_h \,:\, t \subset \hat{\Omega}_j\}$ is the set of fine grid mesh elements making up the subdomain $j$. For subdomains we assume a *finite covering* which is expressed as follows: There exists a constant $k_0$ independent of $p$ such that

$$S_t := \{j \in \{1, \dots, p\} \,:\, t \subset \hat{\Omega}_j\} \quad \text{and} \quad k_0 = \max_{t \in \mathcal{T}_h} \#S_t.$$

Here $S_t$ contains the indices of the subdomains containing element $t$ and $\#S_t$ denotes the number of elements in set $S_t$. The coarse grid and subdomains imply a decomposition of the finite element space $V_h$ defined on $\mathcal{T}_h$ in to the coarse space $V_H$ defined on $\mathcal{T}_H$ and the subdomain spaces $V_{h,j} \subset V_h$ given by $V_{h,j} = \{v \in V_h \,:\, \mathrm{supp}(v) \subset \hat{\Omega}_j\}$.

Then we introduce the notation

$$a_t(u,v) = \int_t (K\nabla u) \cdot \nabla v \, dx, \quad a_{\hat{\Omega}_j}(u,v) = \sum_{t \in \mathcal{T}_{h,j}} \int_t (K\nabla u|_t) \cdot \nabla v|_t \, dx, \quad a_\Omega(u,v) = a(u,v),$$

and define the energy seminorms

$$|u|^2_{a,\omega} = a_\omega(u,u), \quad \forall u \in H^1(\omega),$$

where $\omega$ may be a single element $t$, a subdomain $\hat{\Omega}_j$ or the domain $\Omega$ itself. When it is clear that $u \in H^1_0(\omega)$ then the seminorm becomes a norm and we write $\|.\|_{a,\omega}$ instead and when $\omega = \Omega$ we may omit the domain in the subscript.

After introducing the setting we now come to the formulation of the proposition which is Lemma 2.9 in [*Spillane, Nataf, Dolean, Hauret, Pechstein, Scheichl: Abstract Robust Coarse Spaces for Systems of PDEs via Generalized Eigenproblems in the Overlaps, NuMa-Report No. 2011-07, Johannes Kepler Universität, Linz*].

**Now the proposition to prove reads**: Assume that for each $v \in V_h$ there exists a decomposition into $v = \sum_{j=0}^p v_j$ with $v_0 \in V_H$, $v_j \in V_{h,j}$, $1 \le j \le p$, such that with a constant $C_1 > 0$:

$$\|v_j\|^2_{a,\hat{\Omega}_j} \le C_1 |v|^2_{a,\hat{\Omega}_j} \quad \text{for all } 1 \le j \le p.$$

Then $v = \sum_{j=0}^p v_j$ is a stable splitting with $C_0 = 2 + C_1 k_0 (2k_0 + 1)$.

For the proof proceed in the following steps:

1. Using the assumption of the proposition and the finite covering show

$$\sum_{j=1}^p \|v_j\|^2_{a,\hat{\Omega}_j} \le C_1 k_0 \|v\|^2_a.$$

   Hint: use also $\|u\|^2_a = a(u,u) = \sum_{t \in \mathcal{T}} a_t(u,u)$.

2. Next show for the coarse grid contribution

$$\|v_0\|^2_a \le 2\|v\|^2_a + 2 \left\| \sum_{j=1}^p v_j \right\|^2_a.$$

3. In the next step (this is the most difficult one) show

$$\left\| \sum_{j=1}^p v_j \right\|^2_a \le k_0 \sum_{j=1}^p \|v_j\|^2_{a,\hat{\Omega}_j}.$$

   Hint: start by using $\|u\|^2_a = a(u,u) = \sum_{t \in \mathcal{T}} a_t(u,u)$, use the finite covering assumption for each $t \in \mathcal{T}_h$ and the fact that only a finite number of the $v_j$ are nonzero on $t$. Then employ the inequality $(\sum_{i=1}^m z_i)^2 \le m \sum_{i=1}^m z_i^2$ holding any for $m \in \mathbb{N}$ and numbers $z_i \in \mathbb{R}$.

4. Now combine all intermediate steps to conclude.

**( 12 Points )**