Prof. Dr. Peter Bastian, Linus Seelinger Submission date: 20 June 2019, **6pm (!)**

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Exercise 1 *Coarse grid correction*

In the lecture you have learned the coarse grid correction. In this exercise we are going to derive the algebraic formulation from the variational formulation.
Let $\mathcal{T}_H$ be a triangulation of the domain Ω and let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$. The corresponding the Finite Element bases are denoted by

$$\Phi_H = \{\varphi_i^H \mid i \in \mathcal{I}_H\},$$
$$\Phi_h = \{\varphi_i^h \mid i \in \mathcal{I}_h\}$$

and the Finite Element spaces by

$$V_H = \operatorname{span} \Phi_H,$$
$$V_h = \operatorname{span} \Phi_h.$$

Define the restriction $R_H : V_h \to V_H$ which is represented in the bases Φ_H, Φ_h above by the matrix

$$(R_H)_{ij} = \varphi_i^H(x_j), \quad i \in \mathcal{I}_H, \quad j \in \mathcal{I}_h$$

where the x_j are the Lagrange nodal points: $\varphi_i^h(x_j) = \delta_{ij}$.

Let $u_h^{(k)}$ be given. The coarse grid correction w_H is computed with the variational formulation

$$a(u_h^{(k)} + w_H, v) = l(v) \qquad \forall v \in V_H.$$

1. Show that the following equation holds:

$$\varphi_i^H = \sum_{j \in \mathcal{I}_h} (R_H)_{ij} \varphi_j^h.$$

2. Show the relation

$$R_H A_h (R_H)^T = A_H,$$

with

$$A_h = a(\varphi_j^h, \varphi_i^h),$$
$$A_H = a(\varphi_j^H, \varphi_i^H).$$

3. Derive from the variational formulation the algebraic version presented in the lecture.

(6 Points)

Exercise 2 *Additive Schwarz with and without coarse grid correction*

In this exercise we are going to compare the additive Schwarz method with and without the coarse grid correction.

The implementation of the two parallel solvers for this week's exercise are provided in the directory `uebungen/uebung05`. In this exercise the same Poisson problem is solved as in the previous exercise.

The file `additive_schwarz_exc05.cc` provides the same basic functionality as the additive Schwarz implementation from the previous exercise sheet, but with the following differences:

- It is possible to chose a different length of the domain and a different number of cells in each direction. With this we want to simulate anisotropies of the problem and examine the robustness of the Schwarz methods under such anisotropies.
- The parameters can now be changed by a configuration file called `additive_schwarz.ini`.

The structure of the `ini`-file looks as follows:

```
[domain]
Lx = 1      # length of the domain in x-direction
Ly = 1
Lz = 1

[grid]
dim = 2     # dimension of the problem
nx = 32     # number of cells in x-direction
ny = 32
nz = 4
overlap = 1 # overlap in all directions in decomposition
```

The file `two_level_additive_schwarz_exc05.cc` provides a working implementation of the additive Schwarz method with coarse grid correction. It contains the following parameters

- the length of the domain and the number of cells in each direction
- the desired overlap on the coarse grid
- the desired overlap on the fine grid
- the refinement level L

which can be changed by a second configuration file called `two_level_additive_schwarz.ini`. The structure of the `ini`-file is very similar to the first one:

```
[domain]
Lx = 1       # length of the domain in x-direction
Ly = 1
Lz = 1

[grid]
dim = 2      # dimension of the problem
nx = 32      # number of cells on coarse grid in x-direction
ny = 32
nz = 4
overlapc = 1 # overlap on coarse grid
overlapf = 2 # overlap on fine grid
L = 1        # refinements to obtain fine grid
```

The program uses `YaspGrid` and refines the coarse grid L-times uniformly. The decomposition of the grid on the finest level L corresponds to the subdomains. The original grid is used as a coarse grid. The coarse grid problem is solved on this grid.

Task 1 Have a careful look on the files `two_level_additive_schwarz_exc05.cc` and `two_level_schwarz.hh`. What are the differences to the additive Schwarz method without coarse grid correction? Describe what needs to be done in addition for the two level version.

Task 2 Compare both additive Schwarz methods for different sizes of the overlap on the fine grid, in two and three dimensions with the number of processors $\in \{4, 16, 64\}$. Present the number of iterations in form of a table.

Suggestion in two dimensions: Fix the ratio H/h, i.e. the number of uniform refinements and vary H, i.e. the number of subdomains and the overlap, e.g. $\delta = 1h, 2h, 4h$.

Task 3 Compare both additive Schwarz methods for anisotropic domains in two dimensions. Suggestion: Keep the number of cells the same in both directions (on coarse and fine grid), fix the size of the domain in x-direction and vary the size of the domain in y direction. You may also vary the size of the overlap. Put the number of iterations in a table.

(10 Points)