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Exercise 1 *Localized Stable Splitting*

We consider the two level Schwarz method with coarse grid correction based on the hierarchical construction where $\mathcal{T}_H$ is a coarse mesh of p nonoverlapping subdomains which is uniformly refined to give a fine mesh $\mathcal{T}_h$. Then overlapping subdomains $\hat{\Omega}_j$ are formed by adding elements $t \in \mathcal{T}_h$ from neighboring subdomains. Thus $\mathcal{T}_{h,j} = \{t \in \mathcal{T}_h \,:\, t \subset \hat{\Omega}_j\}$ is the set of fine grid mesh elements making up the subdomain j. For subdomains we assume a *finite covering* which is expressed as follows: There exists a constant k_0 independent of p such that

$$S_t := \{j \in \{1, \dots, p\} \,:\, t \subset \hat{\Omega}_j\} \quad \text{and} \quad k_0 = \max_{t \in \mathcal{T}_h} \#S_t.$$

Here S_t contains the indices of the subdomains containing element t and $\#S_t$ denotes the number of elements in set S_t. The coarse grid and subdomains imply a decomposition of the finite element space V_h defined on $\mathcal{T}_h$ in to the coarse space V_H defined on $\mathcal{T}_H$ and the subdomain spaces $V_{h,j} \subset V_h$ given by $V_{h,j} = \{v \in V_h \,:\, \text{supp}(v) \subset \hat{\Omega}_j\}$.

Then we introduce the notation

$$a_t(u,v) = \int_t (K\nabla u) \cdot \nabla v \, dx, \quad a_{\hat{\Omega}_j}(u,v) = \sum_{t \in \mathcal{T}_{h,j}} \int_t (K\nabla u|_t) \cdot \nabla v|_t \, dx, \quad a_\Omega(u,v) = a(u,v),$$

and define the energy seminorms

$$|u|^2_{a,\omega} = a_\omega(u,u), \quad \forall u \in H^1(\omega),$$

where ω may be a single element t, a subdomain $\hat{\Omega}_j$ or the domain Ω itself. When it is clear that $u \in H^1_0(\omega)$ then the seminorm becomes a norm and we write $\|.\|_{a,\omega}$ instead and when $\omega = \Omega$ we may omit the domain in the subscript.

After introducing the setting we now come to the formulation of the proposition which is Lemma 2.9 in [*Spillane, Nataf, Dolean, Hauret, Pechstein, Scheichl: Abstract Robust Coarse Spaces for Systems of PDEs via Generalized Eigenproblems in the Overlaps, NuMa-Report No. 2011-07, Johannes Kepler Universität, Linz*].

Now the proposition to prove reads: Assume that for each $v \in V_h$ there exists a decomposition into $v = \sum_{j=0}^p v_j$ with $v_0 \in V_H$, $v_j \in V_{h,j}$, $1 \le j \le p$, such that with a constant $C_1 > 0$:

$$\|v_j\|^2_{a,\hat{\Omega}_j} \le C_1 |v|^2_{a,\hat{\Omega}_j} \quad \text{for all } 1 \le j \le p.$$

Then $v = \sum_{j=0}^p v_j$ is a stable splitting with $C_0 = 2 + C_1 k_0(2k_0 + 1)$.

For the proof proceed in the following steps:

1. Using the assumption of the proposition and the finite covering show

$$\sum_{j=1}^p \|v_j\|^2_{a,\hat{\Omega}_j} \le C_1 k_0 \|v\|^2_a.$$

   Hint: use also $\|u\|^2_a = a(u,u) = \sum_{t \in \mathcal{T}} a_t(u,u)$.

2. Next show for the coarse grid contribution

$$\|v_0\|^2_a \le 2\|v\|^2_a + 2 \left\| \sum_{j=1}^p v_j \right\|^2_a.$$

3. In the next step (this is the most difficult one) show

$$\left\| \sum_{j=1}^{p} v_j \right\|_a^2 \le k_0 \sum_{j=1}^{p} \|v_j\|_{a,\hat{\Omega}_j}^2 .$$

   Hint: start by using $\|u\|_a^2 = a(u,u) = \sum_{t \in \mathcal{T}} a_t(u,u)$, use the finite covering assumption for each $t \in \mathcal{T}_h$ and the fact that only a finite number of the v_j are nonzero on t. Then employ the inequality $(\sum_{i=1}^m z_i)^2 \le m \sum_{i=1}^m z_i^2$ holding any for $m \in \mathbb{N}$ and numbers $z_i \in \mathbb{R}$.

4. Now combine all intermediate steps to conclude.

(12 Points)