Parallel Solvers for Finite Elements (WS 2020) **Exercise 9**

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Exercise 1 *Localized Stable Splitting*

We consider the two level Schwarz method with coarse grid correction based on the hierarchical construction where $\mathcal{T}_H$ is a coarse mesh of p nonoverlapping subdomains which is uniformly refined to give a fine mesh $\mathcal{T}_h$. Then overlapping subdomains $\hat{\Omega}_j$ are formed by adding elements $t \in \mathcal{T}_h$ from neighboring subdomains. Thus $\mathcal{T}_{h,j} = \{t \in \mathcal{T}_h \,:\, t \subset \hat{\Omega}_j\}$ is the set of fine grid mesh elements making up the subdomain j. For subdomains we assume a *finite covering* which is expressed as follows: There exists a constant k_0 independent of p such that

$$S_t := \{j \in \{1, \ldots, p\} \,:\, t \subset \hat{\Omega}_j\} \quad \text{and} \quad k_0 = \max_{t \in \mathcal{T}_h} \#S_t.$$

Here S_t contains the indices of the subdomains containing element t and $\#S_t$ denotes the number of elements in set S_t. The coarse grid and subdomains imply a decomposition of the finite element space V_h defined on $\mathcal{T}_h$ in to the coarse space V_H defined on $\mathcal{T}_H$ and the subdomain spaces $V_{h,j} \subset V_h$ given by $V_{h,j} = \{v \in V_h \,:\, \operatorname{supp}(v) \subset \hat{\Omega}_j\}$.

Then we introduce the notation

$$a_t(u,v) = \int_t (K\nabla u) \cdot \nabla v \, dx, \quad a_{\hat{\Omega}_j}(u,v) = \sum_{t \in \mathcal{T}_{h,j}} \int_t (K\nabla u|_t) \cdot \nabla v|_t \, dx, \quad a_\Omega(u,v) = a(u,v),$$

and define the energy seminorms

$$|u|^2_{a,\omega} = a_\omega(u,u), \quad \forall u \in H^1(\omega),$$

where ω may be a single element t, a subdomain $\hat{\Omega}_j$ or the domain Ω itself. When it is clear that $u \in H^1_0(\omega)$ then the seminorm becomes a norm and we write $\|.\|_{a,\omega}$ instead and when $\omega = \Omega$ we may omit the domain in the subscript.

After introducing the setting we now come to the formulation of the proposition which is Lemma 2.9 in [*Spillane, Nataf, Dolean, Hauret, Pechstein, Scheichl: Abstract Robust Coarse Spaces for Systems of PDEs via Generalized Eigenproblems in the Overlaps, NuMa-Report No. 2011-07, Johannes Kepler Universität, Linz*].

Now the proposition to prove reads: Assume that for each $v \in V_h$ there exists a decomposition into $v = \sum_{j=0}^p v_j$ with $v_0 \in V_H$, $v_j \in V_{h,j}$, $1 \le j \le p$, such that with a constant $C_1 > 0$:

$$\|v_j\|^2_{a,\hat{\Omega}_j} \le C_1 |v|^2_{a,\hat{\Omega}_j} \quad \text{for all } 1 \le j \le p.$$

Then $v = \sum_{j=0}^p v_j$ is a stable splitting with $C_0 = 2 + C_1 k_0 (2k_0 + 1)$.

For the proof proceed in the following steps:

1. Using the assumption of the proposition and the finite covering show

$$\sum_{j=1}^p \|v_j\|^2_{a,\hat{\Omega}_j} \le C_1 k_0 \|v\|^2_a.$$

   Hint: use also $\|u\|^2_a = a(u,u) = \sum_{t \in \mathcal{T}} a_t(u,u)$.

2. Next show for the coarse grid contribution

$$\|v_0\|^2_a \le 2\|v\|^2_a + 2 \left\| \sum_{j=1}^p v_j \right\|^2_a.$$

3. In the next step (this is the most difficult one) show

$$\left\| \sum_{j=1}^{p} v_j \right\|_a^2 \le k_0 \sum_{j=1}^{p} \|v_j\|_{a,\hat{\Omega}_j}^2.$$

Hint: start by using $\|u\|_a^2 = a(u,u) = \sum_{t\in\mathcal{T}} a_t(u,u)$, use the finite covering assumption for each $t \in \mathcal{T}_h$ and the fact that only a finite number of the v_j are nonzero on t. Then employ the inequality $(\sum_{i=1}^{m} z_i)^2 \le m \sum_{i=1}^{m} z_i^2$ holding any for $m \in \mathbb{N}$ and numbers $z_i \in \mathbb{R}$.

4. Now combine all intermediate steps to conclude.

(**12 Points**)

Exercise 2 *Solver robustness for diffusion problems with heterogeneous permeability field*

The code for this week's exercise can be found in the directory `uebungen/uebung06`. It provides working implementations of four different parallel solvers, namely

- the additive Schwarz method,

- the additive Schwarz method with coarse grid correction,

- the Multilevel Diagonal Scaling (MDS) method,

- the multiplicative multigrid method.

In this exercise we want to solve the elliptic problem

$$-\nabla \cdot (A(x)\nabla u(x)) = 0 \qquad \text{in } \Omega = (0,1)^d,$$
$$u(x) = \exp(-\|x\|_2^2) \quad \text{on } \partial\Omega.$$

The parameters for this problem are provided in the class `GenericEllipticProblem` in the header file `problem1.hh` whith $A = I_d$ being a constant diffusion tensor as in the previous exercises (I_d being the d-dimensional identity matrix).

In this exercise we will investigate how the different solvers behave under parameter variation.

Task 1 Modify the problem such that the permeability field A is heterogeneous. Implement the diffusion tensor $A = \lambda(x)I_d$ where the scalar $\lambda(x)$ represent the *checkerboard pattern*. Let our square domain Ω be covered by m^2 smaller squares with side length $h_{\text{perm}} = \frac{H}{m}$, see figure 1 for an example with $m = 32$. We subdivide each smaller square into four equal parts and assign a certain value to λ in each part and repeat this pattern over the whole domain Ω.

So λ can take four different values λ_{11}, λ_{12}, λ_{21} and λ_{22}. In the example of figure 1 those values are $\lambda_{11} = 10$, $\lambda_{12} = 10^{-3}$, $\lambda_{21} = 10^3$ and $\lambda_{22} = 0.1$. These values can be changed with the configuration files `additive_schwarz.ini` and `multilevel_settings.ini`.

Implement the checkerboard pattern with arbitrary values λ_{11}, λ_{12}, λ_{21}, λ_{22} for $m = 32$

Task 2 Introduce a new parameter `h_perm` in the ini file and adjust your implementation of task 1 to work for $m = h_{\text{perm}}$. Note: h_{perm} can be chosen independend of the grid size $H = h_0$ but you may assume that $\frac{h_{\text{perm}}}{2}$ is a multiple of h_L.

Task 3 If $\frac{h_{\text{perm}}}{2} < h_l$ for a grid level $l \in \{0,\dots,L\}$ the discretization on this level will produce additional errors as the varying coefficients are not properly resolved by the numerical integration. This may lead to problems for the coarser levels of the multigrid method. We will mitigate this problem by introducing averages into our implementation of task 2.

With the `level()` method of an entity you can get your current grid level. Use this and the information in the ini file to find out if $\frac{h_{\text{perm}}}{2} < h_l$. If this is the case use this information to get an averaged value for the permeability.
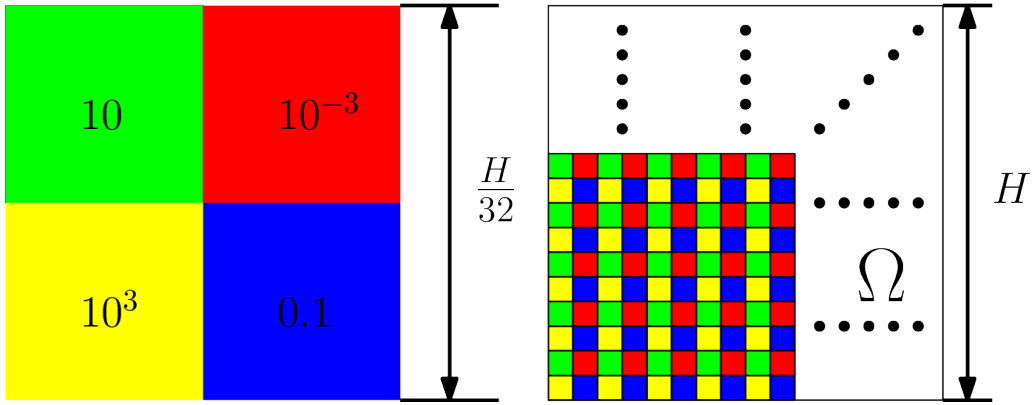
Figure 1: Permeability field in the domain Ω (cube with side length H).

Implement the arithmetic mean, the geometric mean and the harmonic mean and introduce a new parameter in the ini file that allows switching between those.

Task 4 Present the number of iterations for each solver and each averaging technique (in the case of multigrid) for different realizations of the checkerboard pattern in form of a table.

Try the parameter configurations:

- the realization presented in figure 1

- $\lambda_{11} = \lambda_{22}$ and $\lambda_{12} = \lambda_{21}$

- $\lambda_{11} = \lambda_{21} = \lambda_{12} < \lambda_{22}$

Investigate the solvers' behaviour with respect to changing parameter contrast $\frac{\lambda_{12}}{\lambda_{22}}$ and varying h_{perm}. Try different overlaps and number of subdomains.

(**15 Points**)