EXERCISE 1 CROUZEIX-RAVIART A-POSTERIORI ERROR ESTIMATION

Let $\Omega \subset \mathbb{R}^2$ be a domain with Lipschitz boundary and $\{\mathcal{T}_h\}_h$ be a family of conform and shape regular simplex triangulations with maximum edge size h. Let $\mathcal{F}_h$ denote the set of all edges of the mesh $\mathcal{T}_h$. For given $\mathcal{T}_h$ we define following spaces

$$P^1_{c,0} = \{v_h \in C^0(\Omega), \quad v_h\big|_{\partial\Omega} = 0, \quad \forall t \in \mathcal{T}_h : v_h\big|_t \in \mathbb{P}^1\}$$

and

$$P^1_{pt,0} = \{v_h \in L^1(\Omega), \quad \forall t \in \mathcal{T}_h : v_h\big|_t \in \mathbb{P}^1, \quad \forall e \in \mathcal{F}_h : \int_e [\![v_h]\!]_e = 0\},$$

where $[\![f]\!]_e$ is the jump of function f over edge e (if $e \subset \partial\Omega$ we define $f\big|_e = 0$). Only space $P^1_{c,0}$ is a susbspace of $H^1_0(\Omega)$.

For space $V_h := H^1_0(\Omega) + P^1_{pt,0}$ the bilinear form $a_h : V_h \times V_h \to \mathbb{R}$ is defined as

$$a_h(u_h, v_h) = \sum_{t \in \mathcal{T}_h} \int_t \nabla u_h \cdot \nabla v_h dx$$

together with the energy norm

$$\|v_h\|_{V_h} := \sqrt{a_h(v_h, v_h)}.$$

The function $u \in H^1_0(\Omega)$ and $u_h \in V_h$ are solution of

$$\forall v \in H^1_0(\Omega) : \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx.$$

and

$$\forall v_h \in V_h : \sum_{t \in \mathcal{T}_h} \int_t \nabla u_h \cdot \nabla v_h dx = \int_\Omega f v_h dx.$$

Show the a-posteriori error estimation:

$$\|u - u_h\|_{V_h} \leqslant c(\sum_{t \in \mathcal{T}_h} e_t(u_h, f)) + \inf_{v_h \in P^1_{c,0}} \|u_h - v_h\|_{V_h},$$

where the constant c depends only on the shape regularity of the mesh (independent of h) and use

$$e_t(u_h, f) = h_t \|f + \Delta u_h\|_{0,t} + \frac{1}{2} \sum_{e \in \partial t} h_e^{\frac{1}{2}} \|[\![\partial_n u_h]\!]\|_{0,e},$$

with h_e and h_t denoting the length of e and the longest edge in $\mathcal{T}_h$ respectively. *6 points*

EXERCISE 2 CONVECTION-DIFFUSION PROBLEM

In *uebungen/uebung10* of your *dune-npde* module you can find a program that solves a convection-diffusion problem

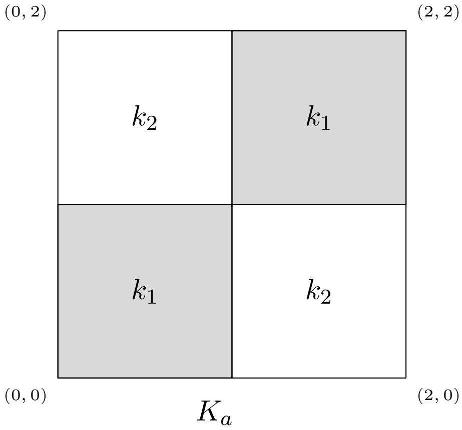
$$
\begin{aligned}
-\nabla \cdot (k(x)\nabla u) + a(x) \cdot \nabla u = 0 \quad & x \in \Omega \\
u(x) = g(x), \quad & x \in \partial\Omega_D \\
(a(x) - k(x)\nabla u) \cdot n = j(x), \quad & x \in \partial\Omega_N
\end{aligned}
$$

using Q^1 and Q^2 finite elements on domain $\Omega = [0,2] \times [0,2] \subset \mathbb{R}^2$.

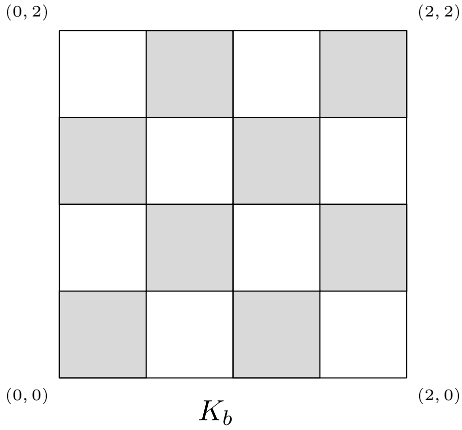
1. First of all we will solve only a diffusion problem ($a(x) = \vec{0}$) with boundary conditions

$$
u = 1 \text{ for } x_1 = 0,\; u = -1 \text{ for } x_1 = 2, \text{ (Dirichlet on the left and on the right side)}
$$
$$
\nabla u \cdot n = 0 \text{ otherwise.}
$$

The permeability field is heterogeneous. In the program we used permeability field K_a. Your task is to implement permeability field K_b, see picture.



K_a



K_b

2. Have a look at the function *flux* (you can find it in the file *utilities.hh*). What does the function compute?

3. Compare the results of *flux* function for K_a and K_b with coefficients $k_1 = 3.10^{-4},\; k_2 = 10^{-1}$ for different refinement level and polynomial degrees. Does it converge to some value?

4. Now we will add advection. Set $a = (1,0)^T$ (parameter $convection = 1$ in *uebung10.ini* and $k_1 = k_2 = 10^{-3}$. What do you observe in the solution for different grid refinements? Can you explain these phenomena?

6 points