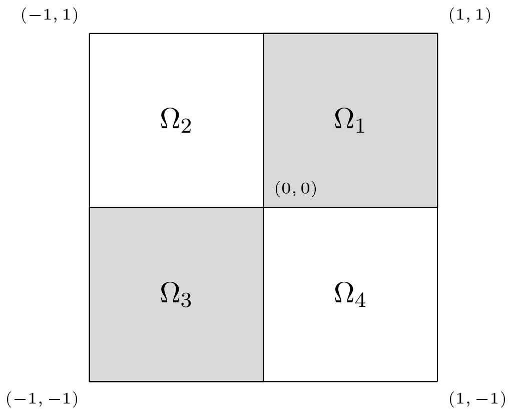
Numerical Solution of Partial Differential Equations, SS 2014 — Exercise Sheet 2
Prof. Peter Bastian — Deadline 7. May 2014
IWR, Universität Heidelberg

EXERCISE 1 ANALYTICAL SOLUTION OF HETEROGENEOUS HEAT EQUATION



On a bounded two-dimensional domain (see picture above) the equation describing stationary heat transfer should be solved:

$$\nabla \cdot (-\lambda \nabla u) = 0, \qquad \forall x \in \Omega, \text{ mit } \Omega = \bigcup_{i=1,\dots,4} \Omega_i,$$

λ is a piecewise constant given by

$$\lambda = \begin{cases} \lambda_1 & x \in \Omega_1 \cup \Omega_3 \\ \lambda_2 & x \in \Omega_2 \cup \Omega_4 \end{cases}.$$

1. Prove, that the following function in polar coordinates

   $$p_i(r, \theta) = r^{\delta}(a_i \sin(\delta\theta) + b_i \cos(\delta\theta))$$

   with constant coefficients $a_i, b_i, \delta \in \mathbb{R}$ in $\Omega \backslash (0,0)$ is harmonical, that means $\Delta p_i = 0$ holds.

2. The functions $p : \Omega \to \mathbb{R}$ is piecewise defined by

   $$p(r, \theta)\big|_{\Omega_i} = p_i(r, \theta), \qquad (i = 1 \dots 4).$$

   Which conditions must be valid at the intersections between subdomains

   $$\Omega_1 \bigcap \Omega_2, \quad \Omega_2 \bigcap \Omega_3, \quad \Omega_3 \bigcap \Omega_4, \quad \Omega_4 \bigcap \Omega_1,$$

   for p to fulfil the physical requirements of the conservation law of the heat transport?

3. *(Bonus)* Determine explicit (using Matlab, Maple, Mathematica or your own programm) the coefficients a_i, b_i, δ for fixed $\delta = 0.5354409455$.

5 (+ 2) points

EXERCISE 2 PROPERTIES OF ENERGY FUNCTIONAL IN DISCRETE SPRING SYSTEM

In the lecture the equation for the total energy stored in the system at state u was derived

$$J^{(n)}(u) = J^{(n)}_{\text{el}} + J^{(n)}_{\text{f}} = \sum_{i=0}^{n} \frac{\kappa_i}{2}(\|u_{i+1} - u_i\|) - l_i)^2 - \sum_{i=1}^{n} u_i \cdot f_i$$

where $J^{(n)} : U \to \mathbb{R}$ and

$$U = \underbrace{\mathbb{R}^3 \times \mathbb{R}^3 \times \cdots \times \mathbb{R}^3}_{n+1 \text{ mal}}.$$

This corresponds to a discrete approximation of the elastic and potential energy (see lecture for details).

Consider $\epsilon \in (0,1)$ with

$$\frac{\epsilon}{2} \sum_{i=0}^{n} \|u_{i+1} - u_i\| \geqslant \sum_{i=0}^{n} l_i.$$

Show that the functional $J^{(n)}(u)$ is bounded below, i.e.

$$\exists C \in \mathbb{R} : J^{(n)}(u) \geqslant C, \quad \forall u \in U. \tag{1}$$

To proof that proceed as follows:

1. At first show that:

$$J^{(n)}_{\text{el}} \geqslant \alpha \left(\sum_{i=0}^{n} \|u_{i+1} - u_i\| \right)^2 + \beta \qquad (\alpha, \beta > 0). \tag{2}$$

2. Futhermore, prove that:

$$\|u\| \leqslant \sqrt{n+2} \left(\sum_{i=0}^{n} \|u_{i+1} - u_i\| + \|u_0\| \right). \tag{3}$$

3. Finally, use both results to show that

$$J^{(n)}_{\text{el}} \geqslant \frac{\alpha^*}{2} \|u\|^2 + \beta^*, \qquad (\alpha^* > 0, \beta^* \in \mathbb{R})$$

The statement (1) is a combination of (2), (3) together with inequality for $J^{(n)}_{\text{f}}(u)$.
Helpful inequality:

$$\left(\sum_{i=1}^{n} a_i \right)^2 \leqslant n \sum_{i=1}^{n} a_i^2$$

10 points

EXERCISE 3 SIMULATION OF DISCRETE SPRING SYSTEM

In this exercise, the solution $u \in \mathbb{R}^{3(n+1)}$ of the discrete energy functional will be determined numerically.

The functional fulfils the inequality

$$J^{(n)}(u) \leqslant J^{(n)}(v) \qquad \forall v \in U.$$

To find a minimum of the functional $J^{(n)}(u)$, the nonlinear algebraic equation

$$\nabla J^{(n)}(u) = 0$$

should be solved.

It holds:

$$\frac{\partial J(u)}{\partial (u_k)_l} = \kappa_{k-1}(\|u_k - u_{k-1}\| - l_{k-1})\frac{(u_k)_l - (u_{k-1})_l}{\|u_k - u_{k-1}\|} + \kappa_k(\|u_{k+1} - u_k\| - l_k)\frac{(u_k)_l - (u_{k+1})_l}{\|u_{k+1} - u_k\|} - (f_k)_l.$$

In *dune-npde* module in directory *dune-npde/uebungen/uebung02* you can find a programm, which is able to compute almost all steps which are necessary to solve the problem.

The nonlinear problem should be solved by an iterative scheme:

$$\frac{\partial J(u^i, u^{i-1})}{\partial (u_k)_l} = \kappa_{k-1}(\|u_k^{i-1} - u_{k-1}^{i-1}\| - l_{k-1})\frac{(u_k^i)_l - (u_{k-1}^i)_l}{\|u_k^{i-1} - u_{k-1}^{i-1}\|} + \kappa_k(\|u_{k+1}^{i-1} - u_k^{i-1}\| - l_k)\frac{(u_k^i)_l - (u_{k+1}^i)_l}{\|u_{k+1}^{i-1} - u_k^{i-1}\|} - (f_k)_l.$$

The iterative scheme starts with an initial value $u^0 \in \mathbb{R}^{3(n+1)}$. In each iteration a linear problem to determine u^i must be solved. Only the functions `assembleMatrix(..)` and `assembleRhs(..)`, which assemble the matrix and the right hand side of the linear problem, need to be implemented properly.

1. Complete the implementation and test it. The programm is configured with the file *uebung02.ini*. The initial values correspond to a silicone-rubber fibre with a cross-section surface of 1 square millimeter. The fibre was stretched to a lengt of 2.5 times the initial length.
2. Test your solution and extend the program in a way that:
   - output contains y-coordinates of the spring-nodes
   - determine the mean and minimum values of y-coordinates
3. *(Bonus):* Do NOT use any *conditionals* in the matrix-iterator loop, that means the instructions which can create some jumps in compiled code (`if`, `switch`, `?:`, `std::max(..)`, etc.).

10 (+ 3) points