Exercise 1 Riesz Theorem (constructive proof)

Let $(V, (.,.)_V)$ be a real Hilbert space and $v' \in V'$ an arbitrary linear form on V. Then there exists a unique $u \in V$ such that

$$\langle v', w\rangle_{V',V} = (u, w)_V \qquad \forall w \in V.$$

Moreover, $\|v'\|_{V'} = \|u\|_V$.

Hints:

1. First prove the uniqueness (under the assumption of an existence) of u.
2. Let denote $M = \{w \in V | \langle v', w\rangle_{V',V} = 0\}$. Show that $M^\perp$ is a one-dimensional subspace of V (or $v' = 0$ holds) and also $V = M \oplus M^\perp$.
3. Show that for $z \in M^\perp$ the vector u is given by

$$u = \frac{\langle v', z\rangle_{V',V}}{\|z\|_V^2} z.$$

When the Riesz Theorem was proved, show the second part:
The map $\tau : V' \to V$ mapping $v' \in V'$ to the corresponding $u \in V$ is linear and an isometry, i.e. $\|\tau v'\|_V = \|v'\|_{V'}$.

6 points

Exercise 2 Analytical Solution For Heat Transfer Equation

Consider the one-dimensional heat transfer equation

$$\partial_t u - \partial_x^2 u = 0$$

in the space-time domain

$$D^+ = \left\{(x,t) \in \mathbb{R}^2 | 0 < x < 1,\ 0 < t < \infty\right\}.$$

1. Show that the initial value problem with initial value $u(x,t)\big|_{t=0} = f$, $f \in C^1([0,1])$ and boundary condition $u(0,t) = u(1,t) = 0$ is solved by

$$u(x,t) = \sum_{n=1}^{\infty} \tilde{f}_n e^{-n^2\pi^2 t} \sin n\pi x,$$

   where $\tilde{f}_n$ denotes the n-th fourier coefficient of f. In order to do this you can use separation of variables $u(x,t) = v(x) \cdot w(t)$.

2. Show that

$$\Phi = \frac{1}{\sqrt{4t}} e^{-\frac{x^2}{4t}}$$

   is a solution of heat trasfer equation. hand side).

5 points

Exercise 3 Heat Transport

Consider the one-dimensional heat transport equation

$$\partial_t u - \partial_x^2 u = 0$$

on the space-time domain

$$D^{+} = \left\{(x,t) \in \mathbb{R}^2 | 0 < x < 1,\ 0 < t < \infty\right\}.$$

The program used to solve this exercise can be found in *dune-npde/uebungen/uebung03* of your *dune-npde* module. It already calculates and prints the fourier coefficients

$$a_n := 2\int_0^1 f(x)\sin 2\pi n x\,dx \qquad b_n := 2\int_0^1 f(x)\cos 2\pi n x\,dx \qquad (N \geqslant n \geqslant 0)$$

of the function

$$f(x,t) = \frac{1}{\sqrt{4t}} e^{-\frac{(x-0.5)^2}{4t}}$$

for time $t_0 = 0.001$. The function `uniformintegration()` is used to determine the necessary integrals. This is allegedly done with a given precision, that can be set with parameter `accuracy`.

1. Describe how the function `uniformintegration()` works. Specify the circumstances under which you can actually estimate the quadrature error with `accuracy`.
2. The quadrature order of local gauss quadrature can be set in the configuration file *uebung03.ini*. Examine the convergence of occuring integrals. Does the convergence behaviour correspond to your expectatitons?
3. Implement a functor realizing the function

$$g(x,t) = \frac{b_0}{2} + \sum_{n=1}^{N} e^{-n^2 4\pi^2 (t-t_0)} \left(a_n \sin n2\pi x + b_n \cos n2\pi x\right).$$

4. Implement a functor that calculates

$$e(t) = \int_0^1 (g(x,t) - f(x,t))^2 dx$$

using `uniformintegration()`. How does $e(t)$ change from $t = 0.001$ to $t = 0.02$? Create vtk files for visualization for time step distance $\Delta t = 0.001$ and explain your observations.

8 points