Exercise 1 Operators on Hilbert space

Let H be a Hilbert space and Y a closed subspace of H. Define the map $P : H \to Y$ for each $v \in H$ as

$$\forall y \in Y : (P(v), y) = (v, y).$$

Let us prove that:

1. Operator P is linear and continuous.
2. For $v \in H$ it holds
$$\|P(v) - v\| = \min_{y \in Y} \|y - v\|$$
(apply *Lax-Milgram Theorem* and *Characterization Theorem*).

5 points

Exercise 2 Projections

Let Y be a subspace of a normed vector space X. An operator $P : X \to X$ is said to be a projection on Y if

$$P^2 = P \quad \text{and} \quad \text{Range}(P) = Y.$$

Show the following:

1. P is a projection if and only if $P : X \to Y$ and $P = I$ on Y.
2. If P is a projection, then $X = \text{Ker}(P) \oplus \text{Range}(P)$, where $\oplus$ denotes a direct sum.
3. Operator P defined in exercise 1 is a projection.

5 points

Exercise 3 Unbounded linear operators

The real trigonometrical polynomials have the form

$$t(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

where $a_n, b_n \in \mathbb{R}$. Let X be the space of all real trigonometrical polynomials on $\Omega = (-\pi, \pi)$ with a finite norm

$$\|t\| = \int_{-\pi}^{\pi} |t(x)| dx.$$

1. Prove, that the derivative $\frac{\partial}{\partial x}$ is a linear operator from X to X.
2. Show, that this operator is not bounded and therefore not continuous.

4 points

EXERCISE 4 H^1-NORM AND HÖLDER-NORM

Let $\Omega = [a,b] \subset \mathbb{R}$. The Hölder-Norm of a real function $f : \Omega \to \mathbb{R}$, $m \in \mathbb{N}, \alpha \in (0,1]$ is defined as

$$\|f\|_{C^{m,\alpha}} := \sum_{|s| \leqslant m} \|\partial^s f\|_\infty + \sum_{|s|=m} \sup\{\frac{|f(x) - f(y)|}{|x-y|^\alpha}; x, y \in \Omega, x \neq y\}$$

Moreover, let $1 < p \leqslant \infty$ and $\alpha := 1 - \frac{1}{p}$.
Prove: Tthere exists a constant $C \in \mathbb{R}$ and $x_0 \in \Omega$, that for $f \in C^1(\Omega)$ it holds:

$$\|f\|_{C^{0,\alpha}} \leqslant |f(x_0)| + C\|f'\|_{L^p}$$

Use Hölder-inequality:
Let $f \in L^p(\Omega), g \in L^q(\Omega)$ and $\frac{1}{p} + \frac{1}{q} = 1$, then it holds $fg \in L^1(\Omega)$ and

$$\|fg\|_{L^1} \leqslant \|f\|_{L^p}\|g\|_{L^q}.$$

3 points

EXERCISE 5 WEAK DIFFERENTIABILITY

Continuous, piecewise-smooth functions in 1D are weakly differentiable (see example 5.31 in the lecture notes). The continuity of the function is crucial.

Consider the function

$$f(x) = \begin{cases} -1 & x \in (-1, 0] \\ 1 & x \in (0, 1) \end{cases}$$

and show that the weak derivative of f does not exist. *2 points*