EXERCISE 1 H^1 FUNCTIONS

Let $\Omega \subset \mathbb{R}^2$ be the unit cube, $\Omega = [0,1] \times [0,1]$.

1. For which α is the function in polar coordinates

$$f(r,\phi) = r^\alpha \, sin(\alpha\phi) \tag{1}$$

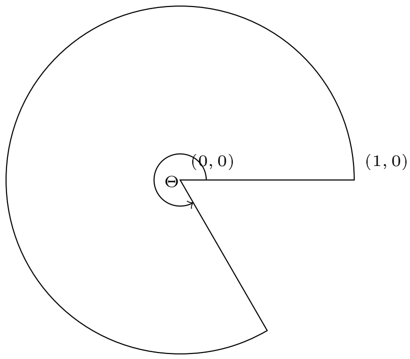
from space $H^1(\Omega)$.

2. The Laplace-Problem $\Delta u = 0$ with pure dirichlet boundary conditions should be solved on a domain Ω (see figure).

The function (1) is a special form of the harmonic function

$$g(r,\phi) = r^{\frac{\pi}{\Theta}} \sin(\frac{\pi}{\Theta}\phi).$$

Show, that g is harmonic, that means $\Delta g = 0$ and write explicit dirichlet boundary condtions.



4 points

EXERCISE 2 DISCONTINUOUS OF H^1-FUNCTIONS IN 2D AND 3D

1. Consider the domain $\Omega = B(0, R) \subset \mathbb{R}^2$, where

$$B(0,R) = \{x \in \mathbb{R}^2 \mid \|x\| < R\}, \quad 0 < R < \frac{1}{e}.$$

   Show in detail that the function

$$f(x) = \ln\left(\ln\left(\frac{1}{r(x)}\right)\right), \quad r(x) = \left(\sum_{i=1}^{2} x_i^2\right)^{\frac{1}{2}}$$

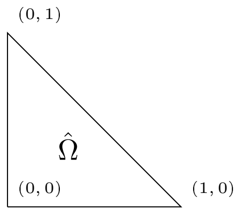
   lies in the space $H^1(\Omega)$ (although it has a singularity in one point).

2. Let $\Omega = B(0, R) \subset \mathbb{R}^3$. In 3D, H^1-functions can have singularities both at isolated points and along one-dimensional curves. Find or construct a function $g = g(x_1, x_2, x_3) \in H^1(\Omega)$, which has a singularity along 1D curve.

   Hint: You can find an inspiration in 1.

6 points

EXERCISE 3 LOCAL PK-BASIS



The local Pk-basis on a d-dimensional simplex (triangle in 2D or tetrahedron in 3D) can be described by polynomials of a maximal degree k. In this exercise, we will restrict ourselfes to a 2D reference element $\hat{\Omega}$

As usual the source code can be found in the directory *uebungen/uebung05/* of the actual *dune-npde* modul. It will be shown (similar to the modul *dune-localfunctions*), how the implementation of local basis can be used both to evaluate the function values and its derivative.

1. Implement a functor, which is able to evaluate a function

   (bestehend aus den Funktionen $(\psi_i)_{i \leq n_k}$ mit $\psi_i \in \mathbb{P}^k(\hat{\Omega})$ und gegebenen Vektor aus Koeffizienten $(\alpha_i)_{i \leq n_k}$ aus $\mathbb{R}$ die Funktion

$$f(x) = \sum_{i=1}^{n_k} \alpha_i \psi_i(x),$$

   where the functions $(\psi_i)_{i \leq n_k}, \psi_i \in \mathbb{P}^k(\hat{\Omega})$ form a Pk-basis and $(\alpha_i)_{i \leq n_k} \in \mathbb{R}$ are the corresponding coefficients of the linear combination. A template to this functor can be found in a file *functors.hh*. You have to implement the function `operator()` in class `LocalFunctor`.

2. After the functor has been already implemented, create $.vtu$ files and visualize the Pk-basis functions. Describe qualitative the characteristic properties of the basis functions.

3. Show, that the Pk-functions really describe a basis of the polynomials with maximal degree k on the reference element. Implement a functor (analog to the previous), which evaluates the monom basis functions and use the functor to proof the linear independence of Pk-functions.

10 points