EXERCISE 1 DOMAIN REGULARITY IN 2D

1. Decide, if the following domains Ω are *Lipschitz-continuous domains*:

   (a)
$$\Omega = \left\{(x, y) \in \mathbb{R}^2 | 0 < x < 1,\ |y| < x^r,\ r > 1\right\}$$

   (b)
$$\begin{aligned}
\Omega_1 &= \left\{(r, \theta) \in \mathbb{R}^2 | 0 < r < 1,\ 0 < \theta < \frac{3}{2}\pi\right\} \\
\Omega_2 &= \left\{(x, y) \in \mathbb{R}^2 | -0.5 < x < 0.5,\ y \geqslant |x|,\ y \leqslant 0.5\right\} \\
\Omega &= \Omega_1 \backslash \Omega_2
\end{aligned}$$

2. Find a domain in 2D, that satisfies a *cone condition* but is not *Lipschitz*.
   Hint: For simply-connected domains, the Lipschitz-continuity is equivalent to the cone condition.

2 points

EXERCISE 2 ROBIN BOUNDARY CONDITIONS

Another frequently used type of natural boundary conditions involves a combination of function values and normal derivatives. Consider the model equation

$$\begin{aligned}
-\nabla \cdot (a_1 \nabla u) + a_0 u &= f \quad \text{in} \quad \Omega, \\
u + \partial_n u &= g \quad \text{on} \quad \partial\Omega,
\end{aligned} \tag{1}$$

where $a_0 = 1, a_1 > 0$, $f \in C(\Omega)$ and $g \in C(\partial\Omega)$. Show, that the solution u of (1) fulfils the weak formulation

$$\int_\Omega (a_1 \nabla u \cdot \nabla v + a_0 uv) + \int_{\partial\Omega} a_1 uv = \int_\Omega fv + \int_{\partial\Omega} a_1 gv \qquad \forall v \in \mathcal{H}^1(\Omega).$$

Show for $f \in \mathcal{H}^1(\Omega)$ that the weak formulation has a unique solution $u \in \mathcal{H}^1(\Omega)$.

Bonus: Prove the uniquenes of solution for the case when $a_0 = 0$. *6 points*

EXERCISE 3 APPROXIMATION ERROR

Let $a : \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega) \to \mathbb{R}$ be a bilinearform $a(u, v) := (\nabla u, \nabla v)$ and $l : \mathcal{H}^1(\Omega)$ be a linear functional. In addition $V_h \subset \mathcal{H}_0^1(\Omega)$ be a finite-dimensional subspace and $u \in \mathcal{H}_0^1(\Omega)$, $u_h \in V_h$ fulfilling

$$a(u, v) = l(v), \quad \forall v \in \mathcal{H}_0^1(\Omega)$$

and

$$a(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h.$$

Show, that

$$\|\nabla u - \nabla u_h\|_0^2 = \|\nabla u\|_0^2 - \|\nabla u_h\|_0^2.$$

3 points

EXERCISE 4 INTERPOLATION

In this exercise you should investigate the property and convergence of interpolation using P_k basis functions. The programm in the directory *uebungen/uebung06* of the actual *dune-npde* modul interpolates a function

$$f(x) = \sum_{i=0}^{d} \frac{1}{x_i + 0.5}$$

in one and two dimensions to P_k space. The interpolation in 1D case is done on an interval $[0, 1]$ and in 2D on an unit triangle as in the previous exercise sheet.

The programm creates VTK files to visualize the reference function f, the interpolated function and the basis functions.

1. Have a look at programm and its structure.
   What happens in the function `interpolate_function()`?

2. The function `uniform_integration()` was changed (in comparison to last exercise). Describe the changes in the function `uniform_integration()` and give the reason for this necessity.

3. To be able to use the function `uniform_integration()` in a right way, one has to use a *dune-pdelab* API constructed interpolation function object `interpolated` together with a class `GridLevelFunction`. Why is this necessary? What would happen otherwise?

4. Run the programm with an init file *uebung06.ini*. The programm computes the L_2 error of the interpolation. Is your observation consistent with your expectation? Estimate (based on programm output) the precision of the L_2 error on level 4 with $k = 4$.

5. *E*xtend the programm in 2D by using a uni square domain Qk basis functions, *Dune::PDELab::QkLocalFiniteElementMap*. Compare the L_2 error of P_1, P_2, Q_1 and Q_2 elements dependent on the number of degrees of freedom. Do you see any differences? Implement an alternative function

   $$g(x) = \begin{cases} 1 & \|x\| < 0.25 \\ 0 & \text{else} \end{cases}.$$

   Plot figures (L_2 error/number of degrees of freedom) of interpolation of f and g using polynoms of degree $1 \leqslant k \leqslant 4$ and explain the difference.

10 points