Exercise 1 Homogeneous Dirichlet problem with $\mathbb{P}^1$ elements

Let $\Omega = [a, b] \subset \mathbb{R}$ be a real 1D domain and $\mathcal{T}_N$ be a equidistant grid on Ω with grid size $h = (b - a)/N$ for $N \in \mathbb{N}$. Let

$$V = \{v \in H^1(\Omega) \mid v(a) = v(b) = 0\}$$

be a vector space and

$$V_h = \{v_h \in \mathbb{C}^0(\Omega) \mid \forall s \in \mathcal{T} : v_h\big|_s \in \mathbb{P}^1(s) \quad \wedge \quad v_h(a) = v_h(b) = 0\}$$

be a finite-dimensional subspace. In addition let l be a continuous linear form $l : V \to \mathbb{R}$ and define a bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx.$$

The vectors $u \in V$ and $u_h \in V_h$ fulfill

$$a(u, v) = l(v), \qquad \forall v \in V$$

and

$$a(u_h, v_h) = l(v_h), \qquad \forall v_h \in V_h.$$

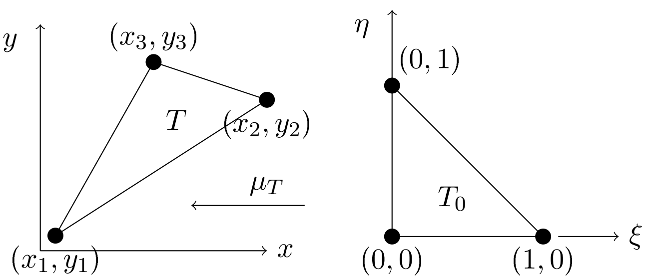
1. Show, that $(\cdot, \cdot)_V = a(\cdot, \cdot)$ induces a scalar product on V.
2. Show, that $u(a + ih) = u_h(a + ih)$ for $i \in 0, \dots, N$.

7 points

Exercise 2 Local $\mathbb{P}^1$ basis on reference element

Let $d = 2$. We consider a unit triangle T_0 with nodes $n_0 = (0, 0)$, $n_1 = (1, 0)$, $n_2 = (0, 1)$ and an arbitrary triangle $T \subset \mathbb{R}^2$ with nodes $a_0 = (x_0, y_0)$, $a_1 = (x_1, y_1)$, $a_2 = (x_2, y_2)$, see picture.

1. The linear function $p \in \mathbb{P}_1(T_0)$ on T_0 can be defined using values in points n_i and the definition is unique. Find a node-basis $\tilde{\varphi}_i,\ i = 1, 2, 3$ of $\mathbb{P}_1(T_0)$ fulfilling $\tilde{\varphi}_i(n_j) = \delta_{ij}$.



2. Find a reference affine mapping $\mu_T : T_0 \to T$. Is this mapping unique and invertible?
3. The functions $\varphi_i,\ i = 1, 2, 3$ are given by

$$\varphi_i(\xi, \eta) := \tilde{\varphi}_i(\mu_T^{-1}(\xi, \eta)).$$

   Prove, that $\varphi \in \mathbb{P}_1(T_0)$ and $\varphi_i(a_j) = \delta_{i,j}$.

4. If you want to integrate a function $vd \in \mathbb{P}_1(T)$ on the T, you can first integrate it on the reference element T_0 (no change in quadrature points) and the result should be modified (regarding original element). Which factor should stay in front of the second integral?

$$\int_T v(x, y) dxdy = \dots \int_{T_0} v\left(\mu_T\left(\xi, \eta\right)\right) d\xi d\eta.$$

5 points

EXERCISE 3 ELLIPTIC OPERATOR IN PDELAB

In this exercise you will solve a PDE for the first time. The programm in the directory *uebungen/uebung08* of the actual *dune-npde* modul solves a Laplace problem

$$-\Delta u(x) = 0 \quad x \in \Omega$$
$$u(x) = g(x) \quad x \in \partial\Omega$$

with Dirichlet boundary conditions using P^k finite elements and conforming triangulation mesh on domain $\Omega = [0,2] \times [0,2] \subset \mathbb{R}^2$.

Your task is to modify the *local operator* to solve a problem

$$\begin{aligned} -\nabla(k(x)\nabla u(x)) &= f(x), \quad x \in \Omega, \\ u(x) &= g(x), \quad x \in \partial\Omega_D, \\ -k(x)\nabla u(x) \cdot n(x) &= j(x), \quad x \in \partial\Omega_N \end{aligned}$$

for scalar functions $k(x), f(x), j(x)$.
Create a VTK-file output and compute L_2-norm of the solution for the situation described below.

1. Choose Dirichlet and Neumann boundary parts of domain $\partial\Omega_D = \{(x,y) | x = 0 \vee x = 2\}$ and $\partial\Omega_N = \partial\Omega \backslash \partial\Omega_D$. Let $g(x,y) = x$ be a function on Dirichlet boundary and $j(x,y) = 0$ be a zero Neumann flux function. The function k is not continuous and is defined by

$$k(x,y) = \begin{cases} 1 & x \leqslant 1 \wedge y > 1 \\ 10^{-5} & x > 1 \wedge y > 1 \\ 1 & x > 1 \wedge y \leqslant 1 \\ 10^{-5} & x \leqslant 1 \wedge y \leqslant 1 \end{cases}$$

   and we have no source term ($f = 0$).

2. Describe (qualitatively) properties of the solution and compare it to the original solution. How does it change if the function j does not disappear, e.g. $j(x,y) = 1$.

3. Describe (qualitatively) properties of the solution if we consider source-term

$$f(x,y) = \exp^{-4((x-1)^2+(y-1)^2)}$$

   and $j(x,y) = 0$.

10 points