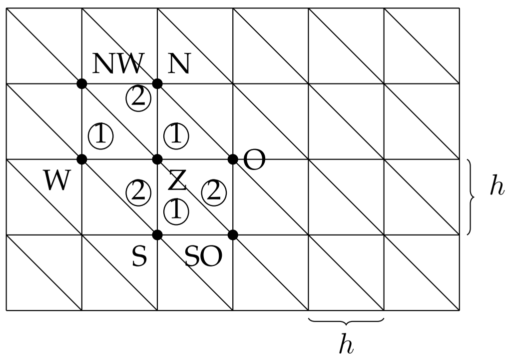
Exercise 1 Stiffness Matrix

We want to solve homogenuous Laplace equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

with P_1 elements on the following grid:



Basis functions of all inner nodes look the same and therefore all rows of the stiffness matrix are identical (except for boundary nodes). As a consequence it is sufficient to look at only one node Z. Let N, O, SO, S, W, NW denote the neighbours of Z.
Determine the matrix values of one row of the stiffness matrix corresponding to a inner node. In oder to do that you have to choose a numeration of basis functions. Express your solution as *finite difference stencil* analogue to finite difference methods.

5 points

Exercise 2 Bramble-Hilbert in 1D

Let $\Omega = [a, b] \subset \mathbb{R}$, $w : \Omega \to \mathbb{R}$ be a function with $w \in H^2(\Omega)$. Let x_k be the vertices of a triangulation of Ω with $x_k = a + \sum_{i=1}^{k} h_i$, $k = 0 \ldots N$ and $h_k > 0$ such that $x_0 = a$ and $x_N = b$ holds. Let v be a piecewise linear interpolation of w fullfilling $v(x_i) = w(x_i)$ for $(i = 0 \ldots N)$. Let $\hat{\Omega} = [0, 1]$ be the reference element and $\mu_k : \hat{\Omega} \to [x_{k-1}, x_k]$ be the corresponding transformation to grid cell $[x_{k-1}, x_k]$.

Show that for $e(x) := w - v$ and $\hat{e}_k(\hat{x}) := e(\mu_k(\hat{x}))$ it holds

$$|\hat{e}_k|_{1,\hat{\Omega}} \leqslant \|\partial_{\hat{x}\hat{x}} \hat{e}_k\|_{0,\hat{\Omega}}$$

and with $h = \max\limits_{1 \leqslant k \leqslant N} \{h_k\}$ it holds

$$\|e\|_{1,\Omega}^2 \leqslant h^2(h+1)\|\partial_{xx} w\|_{0,\Omega}^2.$$

5 points

Exercise 3 Convergence Rates for Poisson Equation

Let $\Omega = [0,a] \times [0,b] \subset \mathbb{R}^2,\ 0 < a, b \in \mathbb{R}$. The Poisson equation

$$-\Delta u(x,y) = \left(\frac{3b}{2}y^2 - \frac{b^2}{2}y - y^3\right)(6x - 3a) + \left(\frac{3a}{2}x^2 - \frac{a^2}{2}x - x^3\right)(6y - 3b), \quad (x,y) \in \Omega \tag{1}$$

with homogenuous Dirichlet boundary condition has the analytical solution

$$u(x,y) = xy(a-x)(b-y)\left(\frac{a}{2} - x\right)\left(\frac{b}{2} - y\right).$$

In *uebungen/uebung09* of your *dune-npde* module you can find a program that solves Poisson equation (1) with P^k finite element on a conform trianglular grid (*UGGrid*) and with Q^k finite element on a conform quadrilateral gride (*YaspGrid*). As domain we chose $\Omega = [0,2] \times [0,2] \subset \mathbb{R}^2$.

1. Implement a method *evaluate* in the class *ExactGradient*. This function should evaluate the gradient of u. Your program can than determine the norms $\|u - u_h\|_{0,\Omega}$ and $\|\nabla(u - u_h)\|_{0,\Omega}$. Extend the program such that it calculates and prints $\|u - u_h\|_{1,\Omega}$ and $\|u - u_h\|_{L_\infty(\Omega)}$ and the corresponding convergence rates.
2. Plot $\|u - u_h\|_{0,\Omega}$, $\|u - u_h\|_{1,\Omega}$ and $\|u - u_h\|_{L_\infty(\Omega)}$ against the number of degrees of freedom for P^k and Q^k elements, $k = 1, 2$. Use logarithmic scale on both axes.
3. Implement a function that calculates

   $$f(u_h, \Omega) = \max_i |u(a_i) - u_h(a_i)|, \text{ where } a_0, \ldots, a_{N-1} \in \overline{\Omega} \text{ are the vertices of our grid.}$$

   What does this function return for P^1 elements? Can you explain your observations?

Hint: Because this problem gets quite large you can compile your code with

make CXXFLAGS='-O3 -march=native -g0 -funroll-loops -ftree-vectorize'

for faster execution.

10 points