Appendix A.

Nabla and Friends

A.1. Notation for Derivatives

The partial derivative

$$\frac{\partial u}{\partial x_i}(x) = \lim_{h\to\infty} \frac{u(x+he_i) - u(x)}{h}$$

of a scalar function $u : \mathbb{R}^n \to \mathbb{R}$ is written in short notation as

$$\partial_{x_i} u(x) = \frac{\partial u}{\partial x_i}(x).$$

Similarly we have for the higher derivatives

$$\partial_{x_i}^2 u(x) = \frac{\partial^2 u}{\partial x_i^2}(x), \qquad \partial_{x_i}\partial_{x_j} u(x) = \frac{\partial^2 u}{\partial x_i \partial x_j}(x), \qquad \dots$$

A vector $\alpha = (\alpha_1, \dots, \alpha_n)^T$ of nonnegative integers α_i is called a *multiindex* of order

$$|\alpha| = \sum_{i=1}^{n} \alpha_i.$$

For a given multiindex α we set

$$\partial^\alpha u(x) = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u(x) = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}(x)$$

For a given nonnegative integer k

$$D^k u(x) = \{\partial^\alpha u(x) \,:\, |\alpha| = k\}$$

denotes the ordered set of all partial derivatives of order k at the point x. Note that $D^k u(x)$ has n^k elements, i.e.$\partial_{x_i}\partial_{x_j} u(x)$ and $\partial_{x_j}\partial_{x_i} u(x)$ are different elements although they have the same value.

For the special cases $k = 1$ and $k = 2$ we identify $D^1 u(x)$ with the gradient $\nabla u(x)$ and $D^2 u(x)$ with the Hessian matrix $\nabla^2 u(x)$ (see below for the definition of gradient and Hessian).

In the case of a function $u(x, y)$, $u : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, we write $D_x^1 u$ or $D_y^2 u$ to indicate the variable with respect to which differentiation is to be applied.

A.2. Vector Differential Calculus

The whole presentation treats the differential operators only in cartesian coordinates.

A.2.1. Nabla Operator

The nabla operator formally is a row or column vector of partial derivatives with respect to all variables of its argument:

$$\nabla = (\partial_1, \ldots, \partial_n)^T \tag{A.1}$$

(when we assume that the argument has n variables).

A.2.2. Gradient

Gradient of a Scalar Nabla applied to a scalar function $u(x_1, \ldots, x_n)$ in n variables gives a vector called "gradient" of the function:

$$\nabla u = (\partial_1 u, \ldots, \partial_n u)^T . \tag{A.2}$$

We can imagine ∇ to be a column vector in this case applied to a scalar which gives a vector.

The gradient of a scalar function in point x is a vector which is perpendicular to the level set $l(c) = \{y : u(y) = c\}$ for $c = u(x)$ pointing in the direction of the steepest increase of the function u.

Gradient of a Vector-valued Function Nabla applied to a vector-valued function

$$u(x) = (u_1(x_1, \ldots, x_n), \ldots, u_m(x_1, \ldots, x_n))^T$$

with m components in n variables gives a matrix called the "Jacobian" of the function:

$$\nabla u = \begin{pmatrix} (\nabla u_1)^T \\ \vdots \\ (\nabla u_m)^T \end{pmatrix} = \begin{pmatrix} \partial_1 u_1 & \ldots & \partial_n u_1 \\ \vdots & & \vdots \\ \partial_1 u_m & \ldots & \partial_n u_m \end{pmatrix} \qquad \text{or} \qquad (\nabla u)_{i,j} = \partial_j u_i. \tag{A.3}$$

If we wish to view the gradient as a column vector and the function u also as acolumn vector (of possibly different size) then we formally have:

$$\text{“}\nabla u\text{”} := \left(\nabla u^T\right)^T . \tag{A.4}$$

Here ∇u^T as an outer product producing a matrix.

In the case of a scalar function u the matrix $\nabla\nabla u = \nabla^2 u$ is called the Hessian matrix.

A.2.3. Divergence

Divergence of a Vector Field The scalar product of nabla with a vector-valued function gives a scalar called the "divergence" of the function:

$$\nabla \cdot u = \sum_{i=1}^{n} \partial_i u_i.$$

Divergence of a Matrix-valued Function The divergence operator applied to a matrix-valued function

$$\sigma(x_1,\ldots,x_n) = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_m \end{pmatrix} = \begin{pmatrix} \sigma_{1,1}(x) & \ldots & \sigma_{1,n}(x) \\ \vdots & & \vdots \\ \sigma_{m,1}(x) & \ldots & \sigma_{m,n}(x) \end{pmatrix}$$

in n variables is defined to yield the divergence for each row of the matrix. Note that σ needs to have as many columns as there are variables. It produces a vector-valued function:

$$\nabla\cdot\sigma = \begin{pmatrix} \nabla\cdot\sigma_1 \\ \vdots \\ \nabla\cdot\sigma_m \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n \partial_j\sigma_{1,j} \\ \vdots \\ \sum_{j=1}^n \partial_j\sigma_{m,j} \end{pmatrix} \qquad \text{or} \qquad (\nabla\cdot\sigma)_i = \sum_{j=1}^n \partial_j\sigma_{i,j}. \tag{A.5}$$

If we regard the divergence as a row vector and σ an $m\times n$ matrix with n also the number of variables, then we can formally write

$$\text{“}\nabla\cdot\sigma\text{”} := \left(\nabla\cdot(\sigma^T)\right)^T. \tag{A.6}$$

Here the inner product $\nabla\cdot(\sigma^T)$ produces a row vector. Note the similarity to the formula (A.4).

A.2.4. Curl

The “curl” (also called “rot”, which is exactly the same thing) of a vector field is defined as

$$\nabla\times u = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix} \tag{A.7}$$

which corresponds to the vector (cross) product $a\times b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)^T$. As stated, it makes only sense for $u:\mathbb{R}\to\mathbb{R}^3$ and there is no obvious extension of the curl operator to n dimensions. However, the related Stokes theorem (see below) can be extended to arbitrary dimensions.

A.2.5. Convection Term in Navier-Stokes Equations

For a vector-valued function u, the convection term in the Navier-Stokes equations is written as $u\cdot\nabla u$ which is formally defined as

$$u\cdot\nabla u = (\nabla u)u = \begin{pmatrix} \nabla u_1\cdot u \\ \vdots \\ \nabla u_n\cdot u \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n u_i\partial_i u_1 \\ \vdots \\ \sum_{i=1}^n u_i\partial_i u_n \end{pmatrix}. \tag{A.8}$$

Note that the scalar product of a vector with a matrix (∇u is a matrix!) is defined as a vector where each component is the scalar multiplication of the vector with a row of the matrix.

A.2.6. Laplacian

Laplacian of a scalar function The Laplacian takes second order derivatives of a scalar function and is defined as

$$\Delta u = \nabla \cdot \nabla u = \sum_{i=1}^{n} \partial_i^2 u. \tag{A.9}$$

Laplace of Vector-valued function The definition of the Laplacian is extended to vector-valued functions by applying it to each component, i.e. the Laplacian of a vector-valued function is again a vector-valued function. In agreement with the conventions above we have:

$$\Delta u = \nabla \cdot \nabla u = \begin{pmatrix} \nabla \cdot \nabla u_1 \\ \vdots \\ \nabla \cdot \nabla u_n \end{pmatrix} = \begin{pmatrix} \Delta u_1 \\ \vdots \\ \Delta u_n \end{pmatrix}. \tag{A.10}$$

A.3. Vector Integral Calculus

A.3.1. Matrix Product

Let T, S be two $m \times n$ matrices, then we define

$$T \,:\, S = \sum_{i=1}^{m} \sum_{j=1}^{n} T_{i,j} S_{i,j}. \tag{A.11}$$

Applied to two vector-valued functions u, v with m components in n variables we have with the definitions from above:

$$\nabla u : \nabla v = \sum_{i=1}^{m} \nabla u_i \cdot \nabla v_i. \tag{A.12}$$

Now let T, S, Q be $n \times n$ matrices. Then the following holds:

$$T : (QSQ^T) = (Q^T T Q) : S. \tag{A.13}$$

This can be shown as follows:

$$\begin{aligned} T : (QSQ^T) &= \sum_{i=1}^{n} \sum_{j=1}^{n} T_{i,j} (e_i^T QSQ^T e_j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} T_{i,j} \left(\sum_{k=1}^{n} Q_{i,k} \left(\sum_{l=1}^{n} S_{kl} Q_{l,j}^T \right) \right) \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} S_{kl} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} T_{i,j} Q_{i,k} Q_{l,j}^T \right) \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} S_{kl} \sum_{i=1}^{n} Q_{k,i}^T \left(\sum_{j=1}^{n} T_{i,j} Q_{j,l} \right) \\ &= (Q^T T Q). \end{aligned}$$

A.3.2. Integration by Parts

Green's formula for sufficiently smooth scalar functions u, v and a suitable bounded domain Ω is

$$\int_\Omega (\partial_i u) v = -\int_\Omega u \partial_i v + \int_{\partial\Omega} uvn_i \tag{A.14}$$

where n_i is the i-th component of the outer unit normal vector n.

For a vector-valued function u and a scalar function v we then have

$$\int_\Omega (\nabla \cdot u) v = -\int_\Omega u \cdot \nabla v + \int_{\partial\Omega} u \cdot n\, v \quad . \tag{A.15}$$

For a matrix-valued function T and a vector valued function v one shows the corresponding formula

$$\int_\Omega (\nabla \cdot T) \cdot v = -\int_\Omega T : \nabla v + \int_{\partial\Omega} (T \cdot n) \cdot v \tag{A.16}$$

which is needed in the variational formulation of the Navier-Stokes equations. Indeed using the definitions above one obtains:

$$\begin{aligned}
\int_\Omega (\nabla \cdot T) \cdot v &= \int_\Omega \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \partial_j T_{i,j} \right) v_i \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ -\int_\Omega T_{i,j} \partial_j v_i + \int_{\partial\Omega} T_{i,j} v_i n_j \right\} \\
&= -\int_\Omega \sum_{i=1}^{n} \sum_{j=1}^{n} T_{i,j} (\nabla v)_{i,j} + \int_{\partial\Omega} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} T_{i,j} n_j \right) v_i \quad .
\end{aligned}$$