Exercise Sheet No.7 Deadline: 10. December 2013

## Exercise 7 Crank Nicholson Time Stepping Scheme

The instationary parabolic problem

$$\begin{aligned} \partial_t T(\vec{x},t) &= \nabla \cdot \left\{ a(\vec{x})\, \nabla T(\vec{x},t) \right\} && \vec{x} \in \Omega \\ T(\vec{x},t) &= 0 && \vec{x} \in \partial\Omega \end{aligned}$$

is known as the heat equation with homogeneous Dirichlet conditions. It describes conductive heat transport for a thermal diffusivity $a$ ($[a] = \frac{m^2}{s}$). Following a common approach we discretize time and space separately. We therefore consider the equation to be an ordinary differential equation in time and employ a time stepping scheme:

For a given $T^0 \in C^2(\Omega)$ find $T^k \in C^2(\Omega)$ :

$$\frac{T^{k+1}(\vec{x}) - T^k(\vec{x})}{\tau} = \nabla \cdot \left\{ a(\vec{x})\, \nabla\big(\sigma\, T^{k+1}(\vec{x}) + (1-\sigma)\, T^k(\vec{x})\big) \right\} \qquad k = 0 \ldots N-1$$

Depending on the value of $\sigma \in [0,1]$ we obtain

- the implicit Euler method ($\sigma = 1$),
- the explicit Euler method ($\sigma = 0$),
- the Crank Nicolson method ($\sigma = 0.5$).

The discretization in space is given by the cell centered finite volume scheme. Hence, in time step $k$ we require

$$\int_E T^{k+1} dV - \tau \int_{\partial E} \sigma\, a\, \vec{n}\, \nabla T^{k+1} dA = \int_E T^k dV + \tau \int_{\partial E} (1-\sigma)\, a\, \vec{n}\, \nabla T^k dA$$

to hold for each cell $E$ of the grid resolving the domain. Furthermore, we require $T^k$ and $a$ to be piecewise constant on each cell and discretize the differential operators accordingly.

Implement a class `HeatConductionAssembler` which assembles the system matrix of a single time step as given above. Its public interface should contain at least the following functions:

- The constructor:

```
HeatConductionAssembler(
    const Grid& grid,
    const SpatialParameters& diffusivity,
    const Sources& sources,
    BoundaryConditions & boundary_conditions
)
```

  We use the class `SpatialParameters` to obtain the thermal diffusivity coefficients.

- A method for assembling the system matrix:

```
void assemble(
    Matrix& A,
    Vector& b,
    const Vector& x,
    const double tau,
    const double sigma
)
```

This function should assemble the system matrix `A` and right hand side `b`. The parameter `sigma` determines the type of the time stepping scheme. The vector `x` should hold the solution of the previous step and `tau` determines the size of the actual step.

Setup a one dimensional test problem for an initial temperature distribution $T^0 \in C^2([0,1])$ given by

$$T^0(x) = \begin{cases} 1\,K & \text{if } x \in [0.4, 0.6]\,m \\ 0\,K & \text{otherwise} \end{cases}$$

and simulate the spreading of the temperature for $t = 0\,s \ldots 0.1\,s$ in a medium with a diffusivity of $a = 1\,\frac{m^2}{s}$.

You may employ the existing grid implementation for the realization of the one dimensional problem by choosing the $y$ dimension of the domain equal to one and enforcing zero Neumann conditions on the boundaries with normal vectors in $y$ direction.

Solve the problem with the Crank Nicolson scheme and a grid resolution of $h = \frac{1}{512}\,m$ and $\tau = h^2$. Assume this solution to be the "true" solution and analyze the grid convergence of all three time stepping schemes for the resolutions

$$h_4 = \frac{1}{256},\ h_3 = \frac{1}{128},\ h_2 = \frac{1}{64},\ h_1 = \frac{1}{32},\ h_0 = \frac{1}{16}$$

and choose

$$\tau_4 = 0.5 \cdot h_4^2 \qquad \text{and} \qquad \tau_i = 2 \cdot \tau_{i+1}$$

in a first analysis and

$$\tau_4 = 128 \cdot h_4^2 \qquad \text{and} \qquad \tau_i = 2 \cdot \tau_{i+1}$$

in a second analysis. In both cases compute the $L_2$- and the $L_\infty$- error of all solutions and thus devise the order of convergence for all three schemes. For a scalar function $f \in L_2(\Omega)$ the $L_2$- and $L_\infty$- error of its approximation $\tilde{f} \in L_2(\Omega)$ are defined as

$$E_2 := \left( \int_\Omega \left(f(x) - \tilde{f}(x)\right)^2 dx \right)^{\frac{1}{2}},$$

$$E_\infty := \sup_{x \in \Omega} \left| f(x) - \tilde{f}(x) \right|.$$

*5 Points*