Exercise Sheet No.9 Deadline: 14. January 2013

EXERCISE 9 CONVECTION DISPERSION EQUATION WITH SLOPE LIMITER

In this exercise you will implement a time explicit discretization scheme for the Convection-Dispersion equation

$$\begin{aligned} \partial_t c + \nabla \cdot \left(-\tilde{D}\, \nabla c + c\, \vec{v} \right) &= 0 && \vec{x} \in \Omega \\ c &= f && \vec{x} \in \partial\Omega_D \\ \vec{n} \cdot \left(-\tilde{D}\, \nabla c + c\, \vec{v} \right) &= J_N && \vec{x} \in \partial\Omega_N \end{aligned}$$

using slope limiters to reduce numerical diffusion. Again, you will implement a cell centered finite volume scheme (cf. last exercise):

In time step k we require for each grid cell E:

$$\int_E c^{k+1} dV - \tau \int_{\partial E \setminus \partial\Omega_N} \vec{n}_E \cdot \tilde{D}\, \nabla c^k \, dA + \tau \int_{\partial E \setminus \partial\Omega_N} \vec{n}_E \cdot \vec{v} \left(c_*^k + \frac{1}{2} \sigma_* \left(h - \tau |\vec{n}_E \cdot \vec{v}| \right) \right) \, dA = \int_E c^k dV - \tau \int_{\partial E \cap \partial\Omega_N} J_N \, dA$$

Here c_*^k denotes the average concentration on the cell adjacent to the face in upwind direction. Let us consider a face e in a two dimensional grid and assume element E at the grid coordinates (i, j) to be the upwind element and element F at the grid coordinates $(i+1, j)$ to be the downwind element. Then we have $c_*^k := c_E^k = c_{(i,j)}^k$.
The slope factor σ_* may be chosen by employing the *MinMod* limiter using

$$\sigma_* = \text{minmod}\left\{ \frac{c_{(i+1,j)}^k - c_{(i,j)}^k}{h}, \frac{c_{(i,j)}^k - c_{(i-1,j)}^k}{h} \right\},$$

$$\text{with} \quad \text{minmod}(a, b) = \begin{cases} a & \text{if} \quad |a| < |b| \wedge a \cdot b > 0, \\ b & \text{if} \quad |a| \geq |b| \wedge a \cdot b > 0, \\ 0 & \text{if} \quad a \cdot b \leq 0, \end{cases}$$

The slope factors for faces with normal vectors in y-direction may be determined analogously.

A program which applies the slope limiter implementation to a simple groundwater test problem is implemented in `main.cc`. Compare the output for the explicit case using slope limiters or just pure upwinding (You can use the SLOPE macro to switch between both cases). The interface of the assembler is given in the files `godunov_assembler.hh` and `godunov_utilities.hh`. In the former of these files, you will have to implement the method `assemble_explicit(...)` such that the solver uses the slope limiter CD transport or not depending on the last argument.

5 Points