EXERCISE 3 ADJACENCY MATRIX

Let I be an index set and $R \subset I \times I$ a symmetric and reflexive relation with

$$\max_i |\{j : (i,j) \in R\}| \leqslant K$$

Define the associated Matrix E via

$$(E)_{ij} := \begin{cases} 1 & (i,j) \in R \\ 0 & \text{sonst} \end{cases}$$

Show that $||E||_2 \leqslant K$.

5 Points

EXERCISE 4 OPTIMAL PARAMETER FOR THE RICHARDSON ITERATION

Let A be a symmetric and positive definite Matrix. The Richardson iteration is given by

$$x_{k+1} = x_k + \omega (b - Ax_k).$$

Assume that the minimal and maximal eigenvalue λ_{min} and λ_{max} of A are known.

1. How can the eigenvalues of the iteration matrix be bounded?
2. Determine the optimal relaxation parameter ω_{opt} and the corresponding spectral radius.

5 Points

EXERCISE 5 DEPENDENCE OF LINEAR SOLVER CONVERGENCE ON THE INITIAL DATA

To obtain the latest software version, go to your `dune-parsolve` directory and type:

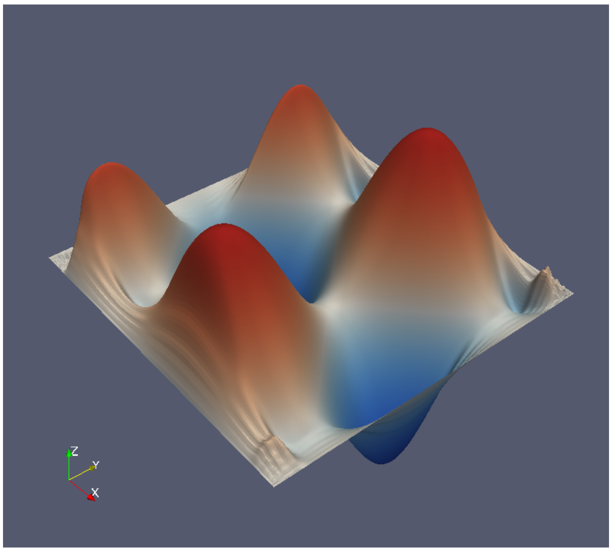
```
$ git stash
$ git pull
$ git stash pop
```

In this exercise we want to consider the Laplace equation with homogeneous Dirichlet boundary conditions

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega = (0,1)^2 \subset \mathbb{R}^2, \\ u &= 0 && \text{auf } \partial\Omega. \end{aligned}$$

The solution obviously reads $u = 0$. We are going to make use of it in order to study the convergence behaviour of the linear solvers by specifying different starting vectors $\neq 0$.

The code skeleton for the exercise is provided in the file `uebungen/uebung02/uebung02.cc`. The program solves the Laplace equation with the starting vector $u_1(x,y)$ defined below doing one Jacobi iteration. The program has been already prepared such that it creates VTK-outputs of the iterates calculated by the linear solvers in order to visualize the distribution of the error with ParaView.



In the file `src/tutorial/istl.cc` of the `dune-parsolve` module you will find some more linear solver and preconditioner objects created. In general, linear iterative methods such as the Jacobi and Gauss-Seidel iteration can be used as stand-alone iterations or as preconditioners. Using them as stand-alone iterations is accomplished by the class `LoopSolver` which simply applies the method in every iteration and checks the convergence criterion.

Modify the file `uebung02.cc` such that it creates VTK-outputs for the following combination of linear solvers, starting vectors and number of iterations:

- Solvers: Jacobi, Gauss-Seidel, Steepest Descent and Conjugate Gradient
- Starting vectors: Given by the functions

$$
\begin{aligned}
u_1(x, y) &= 1, \\
u_2(x, y) &= cos(10x) + sin(10y), \\
u_3(x, y) &= cos(100x) + sin(100y)
\end{aligned}
$$

- Number of iterations: 1, 10 or 100 iterations

Examine the convergence rate of the linear solvers. Does the initial data have an influence on the convergence? *10 Points*