Exercise 6 The parallel Richardson iteration

We want to solve the linear system $Ax = b$ with the Richardson iteration

$$x^{(k+1)} = x^{(k)} + \omega(b - Ax^{(k)}).$$

Let A be the stiffness matrix coming from the discretization of the Poisson equation on the unitsquare with P_1 Finite Elements. We use a structured simplicial mesh with $N = n^2$ degrees of freedom.

To accelerate the computational time we want to do the iteration in parallel with p processors. Therefore we subdivide the unitsquare into p smaller squares and the degrees of freedom are distributed accordingly. With this partitioning we assume that p is a square number such that every processor has $(n/\sqrt{p})^2$ degrees of freedom. The index set of the degrees of freedom is denoted by I, the index set of the i-th processor by I_i. Every processor stores the entries of $x^{(k)}$ corresponding to its degrees of freedom and relevant rows of A.

One iteration of the parallel Richardson iteration consists of the following steps:

- Communication of required entries of $x^{(k)}$ from the neighbouring processors.
- Calculation of $x^{(k+1)}$.

1. Describe the index sets I_i and specify which entries of $x^{(k)}$ the processor i has to communicate with which processor.
2. The computation time for an arbitrary arithmetic operation (addition, subtraction or multiplication) is denoted by t_{op}, the time needed for sending one byte to another processor by t_{byte} and the time needed to set up a message to another processor is denoted by t_{msg}.

   Derive a formula for the total computational time of one iteration with p processors. The entries of $x^{(k)}$ are stored in double precision such that every entry requires 8 byte of memory.

   The formula has to be only asymptotically correct. The matrix rows of the nodes next to the boundary which have less entries can be considered as interiour nodes.
3. Present the speedup of the parallel iteration in a table using the following parameters:

$$\begin{aligned} t_{\text{op}} &= 2 \text{ ns} \\ t_{\text{byte}} &= 20 \text{ ns} \\ t_{\text{msg}} &= 5000 \text{ ns} \\ n &\in \{1024, 4096\} \\ p &\in \{1, 4, 16, 256, 4096\} \end{aligned}$$

12 Points

EXERCISE 7 DOMAIN DECOMPOSITION

In the lecture you have learned the following theorem:
Let $\Omega \subset \mathbb{R}^d$ be Lipschitz domain (open, bounded and connected) and let $f \in L^2(\Omega)$. Then the Poisson problem

$$\begin{aligned} -\Delta u(x) &= f(x) \qquad \forall x \in \Omega \\ u(x) &= 0 \qquad\ \ \forall x \in \partial\Omega \end{aligned} \tag{1}$$

is equivalent to the solution of the two subproblems

$$\begin{aligned} -\Delta u_1(x) &= f(x) && \forall x \in \Omega_1 \\ u_1(x) &= 0 && \forall x \in \partial\Omega_1 \backslash \Gamma \\ u_1(x) &= u_2(x) && \forall x \in \Gamma \\ \frac{\partial u_1(x)}{\partial n_1} &= -\frac{\partial u_2(x)}{\partial n_2} && \forall x \in \Gamma \\ -\Delta u_2(x) &= f(x) && \forall x \in \Omega_2 \\ u_2(x) &= 0 && \forall x \in \partial\Omega_2 \backslash \Gamma \end{aligned} \tag{2}$$

with the non-overlapping decomposition

$$\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}, \quad \Omega_1 \cap \Omega_2 = \varnothing, \quad \Gamma = \partial\Omega_1 \cap \partial\Omega_2, \quad \mu(\partial\Omega_i) > 0,$$

such that the $\partial\Omega_i$ are Lipschitz continuous.

The theory for the Poisson equation can be formulated for right-hand sides $f \in H^{-1}(\Omega)$ as well. But for the general assumption $f \in H^{-1}(\Omega)$, the equivalence (1) $\Leftrightarrow$ (2) does not hold. To see this we are going to consider the following counter example in one dimension for $\Omega = (-1, 1)$:

$$\begin{aligned} -\Delta u(x) &= -2\delta(x) \quad \text{in } \Omega \\ u(-1) &= u(1) = 0 \end{aligned}$$

where $\delta(x)$ denotes the *Dirac delta function*.

1. Find the unique weak solution $u \in H^1(\Omega)$.
2. What are the transmission conditions of $u(x)$ on Γ? Compare them to the transmissions conditions given in (2).
3. What changes in the derivation for the transmission conditions if we take again a right-hand side $f \in L^2(\Omega)$? **Hint:** Cauchy-Schwarz inequality.

8 Points