EXERCISE 12 JACOBI ITERATION AS ADDITIVE SCHWARZ

Let A be the stiffness matrix of a Finite Element discretization with the Finite Element space V_h and basis φ_i^h. We will use the unique representation

$$u_h = \sum_{i=1}^{N_h} x_i \varphi_i^h, \quad x_i \in \mathbb{R}$$

of any function $u_h \in V_h$ as a one dimensional, non-overlapping decomposition of the index sets, i.e. $V_{h,i} = \text{span}\{\varphi_i^h\}$ (cf. exercise 9).

In the lecture you have learned the additive Schwarz iteration

$$x^{(k+1)} = x^{(k)} + \omega \sum_{i=1}^{p} R_i^T A_i^{-1} R_i (b - Ax^{(k)}) \quad \text{with} \quad A_i = R_i A R_i^T. \tag{1}$$

1. Specify how the damping factor ω, the number of subdomains p and the restriction matrices R_i need to be chosen such that (1) describes the Jacobi iteration.

Assume that the following estimate holds:

$$\|x\|_2 \leqslant Ch^{-\frac{d}{2}} \|u_h\|_{0,\Omega}$$

where $\|\cdot\|_2$ denotes the Euclidian norm on $\mathbb{R}^{N_h}$ and C is a constant independent on h.

In order to apply the abstract Schwarz theory to the Jacobi iteration, the following two assumptions need to be fulfilled:

Assumption 1 (Stable splitting)**.**
There exists a constant C_0 such that for all $x \in \mathbb{R}^{N_h}$ there exists a splitting $x = \sum_{i=1}^{p} R_i^T x_i$ such that

$$\sum_{i=1}^{p} \langle R_i^T x_i, R_i^T x_i \rangle_A \leqslant C_0 \langle x, x \rangle_A.$$

Assumption 2 (Strengthened Cauchy-Schwarz inequality)**.**
There exist constants $0 \leqslant \varepsilon_{i,j} \leqslant 1$ for $1 \leqslant i, j \leqslant p$ such that for all x_i and x_j it holds

$$\left|\langle R_i^T x_i, R_j^T x_j \rangle_A\right| \leqslant \varepsilon_{ij} \langle R_i^T x_i, R_i^T x_i \rangle_A^{\frac{1}{2}} \langle R_j^T x_j, R_j^T x_j \rangle_A^{\frac{1}{2}}.$$

2. Show that these assumptions are satisfied by the Jacobi iteration and specify the best possible choice of the constants C_0 and ε_{ij}.

8 Points

EXERCISE 13 VARIANT OF STABLE SPLITTING

We consider the two level Schwarz method with coarse grid correction based on the hierarchical construction where $\mathcal{T}_H$ is a coarse mesh of p nonoverlapping subdomains which is uniformly refined to give a fine mesh $\mathcal{T}_h$. Then overlapping subdomains $\hat{\Omega}_j$ are formed by adding elements $t \in \mathcal{T}_h$ from neighboring subdomains. Thus $\mathcal{T}_{h,j} = \{t \in \mathcal{T}_h \, : \, t \subset \hat{\Omega}_j\}$ is the set of fine grid mesh elements making up the subdomain j. For subdomains we assume a *finite covering* which is expressed as follows: There exists a constant k_0 independent of p such that

$$S_t := \{j \in \{1, \ldots, p\} \, : \, t \subset \hat{\Omega}_j\} \quad \text{and} \quad k_0 = \max_{t \in \mathcal{T}_h} \#S_t.$$

Here S_t contains the indices of the subdomains containing element t and $\#S_t$ denotes the number of elements in set S_t. The coarse grid and subdomains imply a decomposition of the finite element space V_h defined on $\mathcal{T}_h$ in to the coarse space V_H defined on $\mathcal{T}_H$ and the subdomain spaces $V_{h,j} \subset V_h$ given by $V_{h,j} = \{v \in V_h \,:\, \text{supp}(v) \subset \hat{\Omega}_j\}$.

Then we introduce the notation

$$a_t(u, v) = \int_t (K\nabla u) \cdot \nabla v \, dx, \quad a_{\hat{\Omega}_j}(u, v) = \sum_{t \in \mathcal{T}_{h,j}} \int_t (K\nabla u|_t) \cdot \nabla v|_t \, dx, \quad a_\Omega(u, v) = a(u, v),$$

and define the energy seminorms

$$|u|^2_{a,\omega} = a_\omega(u, u), \quad \forall u \in H^1(\omega),$$

where ω may be a single element t, a subdomain $\hat{\Omega}_j$ or the domain Ω itself. When it is clear that $u \in H^1_0(\omega)$ then the seminorm becomes a norm and we write $\|.\|_{a,\omega}$ instead and when $\omega = \Omega$ we may omit the domain in the subscript.

After introducing the setting we now come to the formulation of the proposition which is Lemma 2.9 in [*Spillane, Nataf, Dolean, Hauret, Pechstein, Scheichl: Abstract Robust Coarse Spaces for Systems of PDEs via Generalized Eigenproblems in the Overlaps, NuMa-Report No. 2011-07, Johannes Kepler Universität, Linz*].

Now the proposition to prove reads: Assume that for each $v \in V_h$ there exists a decomposition into $v = \sum_{j=0}^{p} v_j$ with $v_0 \in V_H$, $v_j \in V_{h,j}$, $1 \leqslant j \leqslant p$, such that with a constant $C_1 > 0$:

$$\|v_j\|^2_{a,\hat{\Omega}_j} \leqslant C_1 |v|^2_{a,\hat{\Omega}_j} \quad \text{for all } 1 \leqslant j \leqslant p.$$

Then $v = \sum_{j=0}^{p} v_j$ is a stable splitting with $C_0 = 2 + C_1 k_0(2k_0 + 1)$.

For the proof proceed in the following steps:

1. Using the assumption of the proposition and the finite covering show

$$\sum_{j=1}^{p} \|v_j\|^2_{a,\hat{\Omega}_j} \leqslant C_1 k_0 \|v\|^2_a.$$

   Hint: use also $\|u\|^2_a = a(u, u) = \sum_{t \in \mathcal{T}} a_t(u, u)$.

2. Next show for the coarse grid contribution

$$\|v_0\|^2_a \leqslant 2\|v\|^2_a + 2 \left\| \sum_{j=1}^{p} v_j \right\|^2_a.$$

3. In the next step (this is the most difficult one) show

$$\left\| \sum_{j=1}^{p} v_j \right\|^2_a \leqslant k_0 \sum_{j=1}^{p} \|v_j\|^2_{a,\hat{\Omega}_j}.$$

   Hint: start by using $\|u\|^2_a = a(u, u) = \sum_{t \in \mathcal{T}} a_t(u, u)$, use the finite covering assumption for each $t \in \mathcal{T}_h$ and the fact that only a finite number of the v_j are nonzero on t. Then employ the inequality $(\sum_{i=1}^{m} z_i)^2 \leqslant m \sum_{i=1}^{m} z_i^2$ holding any for $m \in \mathbb{N}$ and numbers $z_i \in \mathbb{R}$.

4. Now combine all intermediate steps to conclude.

12 Points

EXERCISE 14 ADDITIVE SCHWARZ WITH AND WITHOUT COARSE GRID CORRECTION

In this exercise we are going to compare the additive Schwarz method with and without the coarse grid correction. As in the previous exercises, go to your `dune-parsolve` directory and type

```
$ git stash
$ git pull
$ git stash pop
```

to obtain the latest software version. The implementation of the two parallel solvers for this week's exercise are provided in the directory `uebungen/uebung05`. In this exercise the same Poisson problem is solved as in the previous exercise.

The file `additive_schwarz_exc05.cc` provides the same basic functionality as the additive Schwarz implementation from the previous exercise sheet, but with the following differences:

- It is possible to chose a different length of the domain and a different number of cells in each direction. With this we want to simulate anisotropies of the problem and examine the robustness of the Schwarz methods under such anisotropies.
- The parameters can now be changed by a configuration file called `additive_schwarz.ini`.

The structure of the `ini`-file looks as follows:

```
[domain]
Lx = 1        # length of the domain in x-direction
Ly = 1
Lz = 1

[grid]
dim = 3       # dimension of the problem
nx = 32       # number of cells in x-direction
ny = 32
nz = 32
overlap = 1 # overlap in all directions in decomposition
```

The file `two_level_additive_schwarz_exc05.cc` provides a working implementation of the additive Schwarz method with coarse grid correction. It contains the following parameters

- the length of the domain and the number of cells in each direction
- the desired overlap on the coarse grid
- the desired overlap on the fine grid
- the refinement level L

which can be changed by a second configuration file called `two_level_additive_schwarz.ini`. The structure of the `ini`-file is very similar to the first one:

```
[domain]
Lx = 1         # length of the domain in x-direction
Ly = 1
Lz = 1

[grid]
dim = 2        # dimension of the problem
nx = 32        # number of cells on coarse grid in x-direction
ny = 32
nz = 4
overlapc = 1 # overlap on coarse grid
overlapf = 2 # overlap on fine grid
L = 1          # refinements to obtain fine grid
```

The program uses `Yasp-Grid` and refines the coarse grid L-times uniformly. The decomposition of the grid on the finest level L corresponds to the subdomains. The original grid is used as a coarse grid. The coarse grid problem is solved on this grid.

Task 1 Have a careful look on the files `two_level_additive_schwarz_exc05.cc` and `two_level_schwarz.hh`. What are the differences to the additive Schwarz method without coarse grid correction? Describe what needs to be done in addition for the two level version.

Task 2 Compare both additive Schwarz methods for different sizes of the overlap on the fine grid, in two and three dimensions with the number of processors $\in \{4, 16, 64\}$. Present the number of iterations in form of a table.

Suggestion in two dimensions: Fix the ratio H/h, i.e. the number of uniform refinements and vary H, i.e. the number of subdomains and the overlap, e.g. $\delta = 1h, 2h, 4h$.

Task 3 Compare both additive Schwarz methods for anisotropic domains in two dimensions. Suggestion: Choose keep the number of cells the same in both directions (on coarse and fine grid), fix the size of the domain in x-direction and vary the size of the domain in y direction. You may also vary the size of the overlap. Put the number of iterations in a table.

10 Points