Exercise 15 Parallel Multigrid

In the lecture the restrictions $r_{l,i}$, $R_{l,i}$ and R_l have been introduced. With $r_{l,i} : \mathbb{R}^{I_l} \to \mathbb{R}^{I_{l,i}}$ we denote the restriction to the subdomain i, such that for $x_l \in \mathbb{R}^{I_l}$ it holds

$$(r_{l,i} x_l)_j = (x_l)_j \quad \forall j \in I_{l,i}$$

as in the Schwarz methods. The multilevel restriction $R_l : \mathbb{R}^{I_{l+1}} \to \mathbb{R}^{I_l}$ is defined as

$$(R_l x_{l+1})_\alpha = \sum_{\beta \in I_{l+1}} \theta^{l,l+1}_{\alpha,\beta} (x_{l+1})_\beta$$

for $x_{l+1} \in \mathbb{R}^{I_{l+1}}$. The restriction of R_l to the subdomain i is denoted by $R_{l,i} : \mathbb{R}^{I_{l+1,i}} \to \mathbb{R}^{I_{l,i}}$ and is defined for $x_{l+1,i} \in \mathbb{R}^{I_{l+1,i}}$ as

$$(R_{l,i} x_{l+1,i})_\alpha = \sum_{\beta \in I_{l+1,i}} \theta^{l,l+1}_{\alpha,\beta} (x_{l+1,i})_\beta .$$

In this exercise we consider additional properties of these operators besides Observation 6.4 and Observation 6.5 in the lecture notes.

1. Show that the following equality does **not** hold in general,

$$R_{l,i}\, r_{l+1,i}\, x_{l+1} = r_{l,i}\, R_l\, x_{l+1}. \tag{1}$$

   Hint: It is sufficient to consider this in one dimension.

2. Let $\hat{I}_{l,i} \subset I_{l,i}$ have the properties: $\alpha \in \hat{I}_{l,i} \Rightarrow s_a \in \Omega_i \wedge s_a \notin \partial\Omega_i$. Then (1) holds $\forall \alpha \in \hat{I}_{l,i}$.

3. Describe the consequences implied by these properties for the implementation of **overlapping** multigrid methods.

8 Points

EXERCISE 16 TRANSFORMATION BETWEEN LAGRANGE AND HIERARCHICAL BASIS

Let a 1D coarse grid with N elements of width H be given. The finer grids of width $\frac{H}{2^l}$ are generated through uniform refinement. On these grids it is possible to use both the standard basis and the hierarchical basis. See figure 1 for a representation of the bases based on a coarse grid consisting of 2 elements.

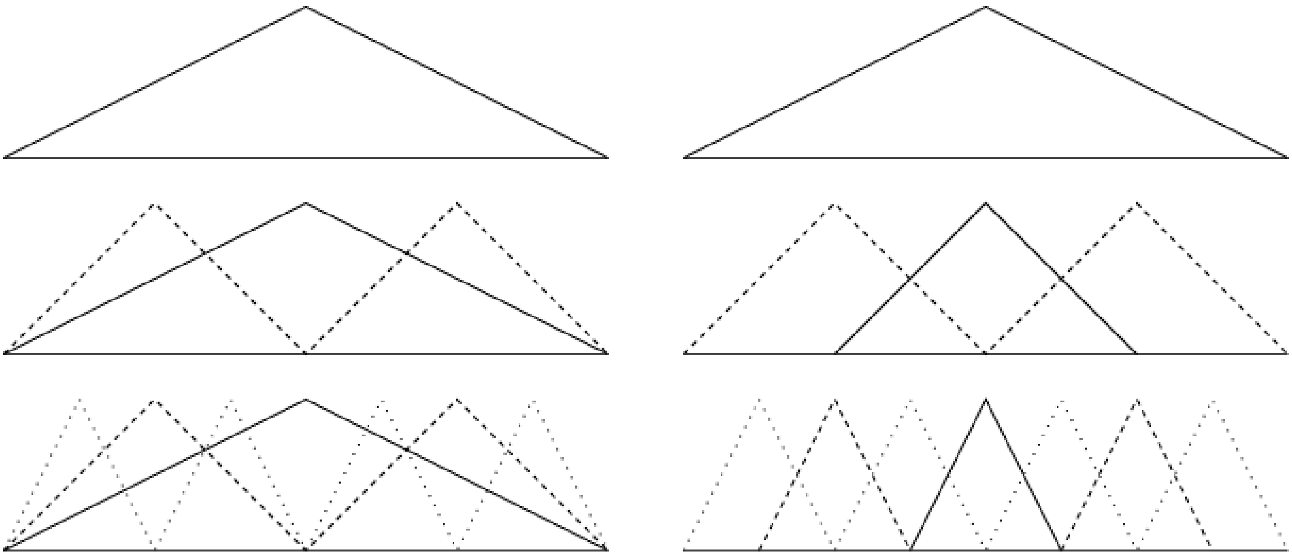

Abbildung 1: Hierarchical basis (left) versus standard nodal basis (right) in 1D

Calculate the transformation between these two bases on the grid level l. *7 Points*

EXERCISE 17 SOLVER ROBUSTNESS FOR DIFFUSION PROBLEMS WITH HETEROGENEOUS PERMEABILITY FIELD

As in the previous exercises, go to your `dune-parsolve` directory and type

```
$ git stash
$ git pull
$ git stash pop
```

to obtain the latest software version. The code for this week's exercise can be found in the directory `uebungen/uebung06`. It provides working implementations of four different parallel solvers, namely

- the additive Schwarz method,
- the additive Schwarz method with coarse grid correction,
- the Multilevel Diagonal Scaling (MDS) method,
- the multiplicative multigrid method.

In this exercise we want to solve the elliptic problem

$$-\nabla \cdot (A(x)\nabla u(x)) = 0 \qquad \text{in } \Omega = (0,1)^d,$$
$$u(x) = \exp(-\|x\|_2^2) \qquad \text{on } \partial\Omega.$$

The parameters for this problem are provided in the class *GenericEllipticProblem* in the header file `problem1.hh` where $A = I_d$ as in the previous exercises. Purpose of this exercise is to investigate the robustness of the solvers under anisotropies coming from a space-dependent diffusion tensor.

Task 1 Modify the problem such that the permeability field A is heterogeneous. The space-dependent scalar $\lambda(x)$ in the diffusion tensor should represent the *checkerboard pattern*, thus it can take the four values $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}$ in general. These values can be changed with the configuration files `additive_schwarz.ini` and `multilevel_settings.ini`.

Implement the checkerboard pattern with arbitrary values $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}$ as presented in figure 2. **Note** that figure 2 shows the case $\lambda_{11} = 10, \lambda_{12} = 10^{-3}, \lambda_{21} = 10^3, \lambda_{22} = 0.1$.

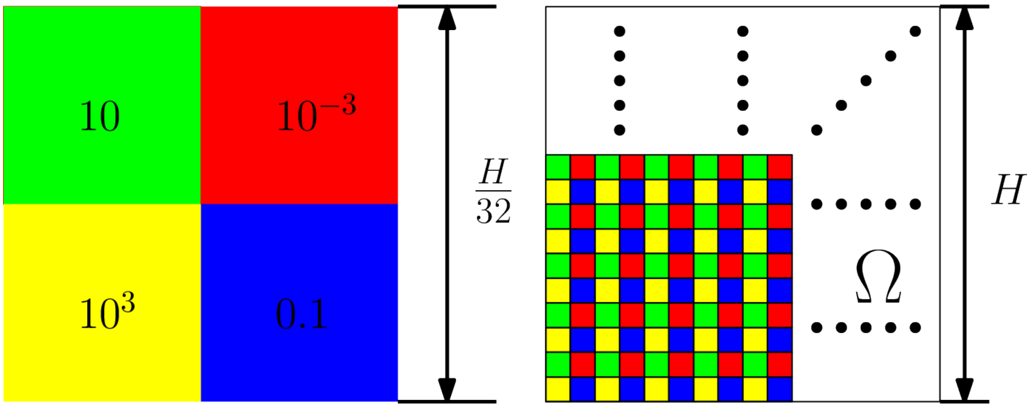

Abbildung 2: Permeability field in the domain Ω (cube with side length H).

Task 2 Present the number of iterations for each solver for various realizations of the checkerboard pattern in form of a table. Suggestions:

- $\lambda_{11} = \lambda_{21}, \lambda_{12} = \lambda_{22}$ and $\lambda_{11} = \lambda_{22}, \lambda_{12} = \lambda_{21}$ for different ratios of $\frac{\lambda_{11}}{\lambda_{12}}$
- the realization presented in figure 2

Do the calculations on a fine grid of the size 512×512 with number of processors $\in \{1, 4, 16, 64\}$. Choose a coarse grid of the size 32×32 and 64×64 and vary also the size of the overlap.

15 Points